ATME College of Engineering

Department of EEE
Emitting Elite Energy

Control Systems

Module 1: Classification of Control System and Mathematical Modeling of Physical Systems

Course Credit Description

Sl. No.	Course Code	Course Type	Course	Teaching Hours /Week	Exam Duration in Hours	CIE	SEE	Total Marks	Credits
1	BEE602	Core	Control Systems	4	3	50	50	100	4

Course Objectives

- To demonstrate mathematical modeling of control systems.
- To obtain transfer function of systems through block diagram manipulation and reduction
- To use Mason’s gain formula for finding transfer function of a system
- To discuss transient and steady state time response of a simple control system.
- To discuss the stability of linear time invariant systems and Routh - Hurwitz criterion
- To investigate the trajectories of the roots of the characteristic equation when a system parameter is varied.
- To conduct the control system analysis in the frequency domain.
- To analyze stability of a control system using Nyquist plot.
- To discuss stability analysis using Bode plots.

Control System Course Contents

Module – 1	
Introduction to control systems: Introduction, classification of control systems. Mathematical models of physical systems: Modelling of mechanical system elements, electrical systems, Analogous systems, Transfer function, Single input single output systems, Procedure for deriving transfer functions, servomotors, synchros, gear trains. (10 Hours)	
Bloom's Taxonomy Level	L1 – Remembering, L2 – Understanding, L3 – Applying. L4 – Analysing

Module1 presents a brief introduction to control systems, control system types and classification and a few recent developments in control system, and illustrates the process of deriving the Mathematical Models of Physical system and the concept of transfer function of some important control systems

Control System Course Contents

Module - 2	
Block diagram: Block diagram of a closed loop system, procedure for drawing block diagram and block diagram reduction to find transfer function. Signal flow graphs: Construction of signal flow graphs, basic properties of signal flow graph, signal flow graph algebra, construction of signal flow graph for control systems. (10 Hours)	
Bloom’s Taxonomy Level	L1 – Remembering, L2 – Understanding, L3 – Applying. L4 – Analysing.

Module 2 presents a Block diagram representation of control system and their reduction techniques. And presents a Signal flow graph representation of control system and use of Mason's gain formula for finding overall gain of control system.

Control System Course Contents

Module - 3	
Time Domain Analysis: Standard test signals, time response of first order systems, time response of second order systems, steady state errors and error constants, types of control systems. Routh Stability criterion: BIBO stability, Necessary conditions for stability, Routh stability criterion, difficulties in formulation of Routh table, application of Routh stability criterion to linear feedback systems, relative stability analysis. (10 Hours)	
Bloom’s Taxonomy Level	L1 – Remembering, L2 – Understanding, L3 – Applying. L4 – Analysing.

Module 3 deals with the Time response analysis, which includes steady state and transient response of the control system and presents Error analysis and Stability analysis through Routh’s Hurwitz criteria

Course Contents

Module - 4	
Root locus technique: Introduction, root locus concepts, construction of root loci, rules for the construction of root locus. Frequency Response analysis: Co-relation between time and frequency response – 2nd order systems only. Bode plots: Basic factors G(iw)/H(jw), General procedure for constructing bode plots, computation of gain margin and phase margin. (10 Hours)	
Bloom's Taxonomy Level	L1 – Remembering, L2 – Understanding, L3 – Applying, L4 – Analysing

Module 4 gives Root locus plots for stability analysis and on Stability analysis in Frequency domain through Bode plots for stability analysis

Course Contents

Module - 5	
Nyquist plot: Principle of argument, Nyquist stability criterion, assessment of relative stability using Nyquist criterion. Design of Control Systems: Introduction, Design with the PD Controller, Design with the PI Controller, Design with the PID Controller, Design with Phase-Lead Controller, Design with Phase - Lag Controller, Design with Lead-Lag Controller. (10 Hours)	
Bloom's Taxonomy Level	L1 – Remembering, L2 – Understanding, L3 – Applying, L4 – Analysing

Module 5 on Stability analysis in Frequency domain through Polar plots (Nyquist plots) and types of compensators, which are added to a control system to compensate for undesirable behavior

Course Outcomes

At the end of the course the students will be able to:

- Analyse the modelling of mechanical and electrical systems and develop the transfer functions of the control systems. [L4]
- Analyse and develop the transfer function of the system by using block diagram reduction technique and signal flow graph. [L4]
- Analyse the time response of first order and second order system and determine the stability of system using RH criteria. [L4]
- Analyse the stability of the system using Root Locus and Bode plot. [L4]
- Analyse the stability of the system using nyquist plot and design the controllers and compensators.[L4]

Text Books and Reference Books

1. Control Systems by Anand Kumar PHI 2nd Edition.

	Reference Book Name	Author Name	Publisher	Edition
1	Automatic Control Systems	Farid Golnaraghi, Benjamin C. Kuo	Wiley	9th Edition, 2010
2	Control Systems Engineering	Norman S. Nise	Wiley	4th Edition, 2004
3	Modern Control Systems	Richard C Dorf et al	Pearson	11th Edition, 2008
4	Control Systems, Principles and Design	M.Gopal	McGaw Hill	4th Edition, 2012
5	Control Systems Engineering	S. Salivahanan et al	Pearson	1st Edition, 2015

- Engineers create products that help people. Our quality of life is sustained and enhanced through engineering. To accomplish this, engineers strive to understand, model, and control the materials and forces of nature for the benefit of humankind.

- A key area of engineering that reaches across many technical areas is the multidisciplinary field of control system engineering.

- Control engineers are concerned with understanding and controlling segments of their environment, often called **systems**, which are interconnections of elements and devices for a desired purpose

- The system might be something as clear-cut as an automobile cruise control system, or as extensive and complex as a direct brain-to-computer system to control a manipulator.

- Control engineering deals with the design (and implementation) of control systems using linear, time-invariant mathematical models representing actual physical nonlinear, time-varying systems with parameter uncertainties in the presence of external disturbances

- A **sensor** is a device that provides a measurement of a desired external signal.
- For example, resistance temperature detectors (RTDs) are sensors used to measure temperature.
- An **actuator** is a device employed by the control system to alter or adjust the environment.
- An electric motor drive used to rotate a robotic manipulator is an example of a device transforming electric energy to mechanical torque
- A challenge for control engineers today is to be able to create simple, yet reliable and accurate mathematical models of many of our modern, complex, interrelated, and interconnected systems

Control system engineering focuses on the modeling of a wide assortment of physical systems and using those models to design controllers that will cause the closed-loop systems to possess desired performance characteristics, such as stability, relative stability, steady-state tracking with prescribed maximum errors, transient tracking (percent overshoot, settling time, rise time, and time to peak), rejection of external disturbances, and robustness to modeling uncertainties.

The extremely important step of the overall design and implementation process is designing the control systems, such as PID controllers, lead and lag controllers, state variable feedback controllers, and other popular controller structures

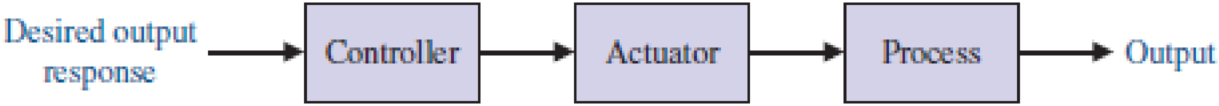
A control system consists of interconnected components to achieve a desired purpose.

A **control** system is an interconnection of components forming a system configuration that will provide a desired system response

Therefore a component or process to be controlled can be represented by a block, as shown in Figure

FIGURE 1.1
Process to be controlled.

FIGURE 1.2
Open-loop control system (without feedback).



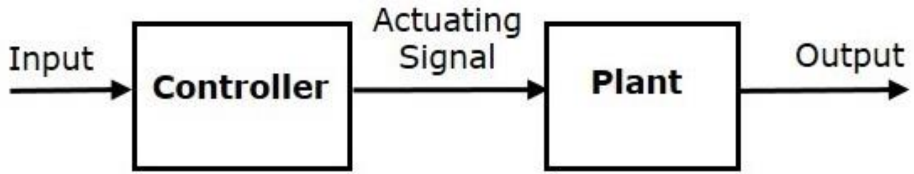
An **open-loop control system** uses a controller and an actuator to obtain the desired response, as shown in Figure

An open-loop system is a system without feedback.

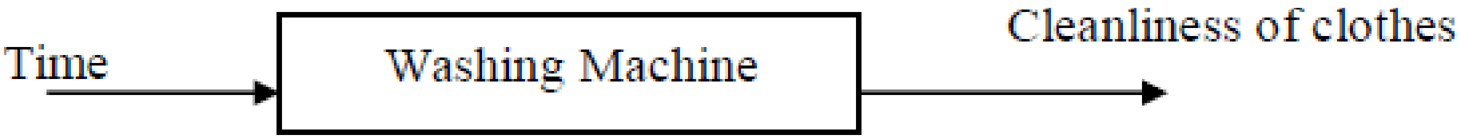
An open-loop control system utilizes an actuating device to control the process directly without using feedback

Open loop control system

❖ Open loop system are those without feedback and the output has no effect on control action. The block diagram of open loop control system is shown



Domestic washing machine typical example of an open loop system. In this case, the preset washing cycle activates various operations for a set period of time. By this preset sequence of operations, the clothes are expected to emerge clean. It is a open loop control system as the cleanliness of cloth is not measured and no corrective action is taken during operations



A common example of an open-loop control system is a microwave oven set to operate for a fixed time.

OPEN LOOP CONTROL SYSTEM

The open loop control system is also known as control system without feedback or non-feedback control systems. In open loop systems the control action is independent of the desired output. In this system the output is not compared with the reference input.

The component of the open loop systems are controller and controlled process. The controller may be amplifier, filter etc. depends upon the system. An input is applied to the controller and the output of the controller gives to the controlled process and we get the output (desired).

Examples:

1. Automatic washing machine is the example of the open loop systems.

   In the machine the operating time is set manually. After the completion of set time the machine will stops, with the result we may or may not get the desired (output) amount of cleanliness of washed cloths because there is no feedback is provided to the machine for desired output.

2. Immersion rod is another example of open loop system. The rod heats the water but how much heating is required is not sense by the rod because of no feedback to the rod.

3. A field control d.c. motor is the example of open loop system.

4. For automatic control of traffic the lamps of three different colours (red, yellow and green) are used. The time for each lamp is fixed. The operation of each lamp does not depends upon the density of the traffic but depends upon the fixed time. Thus, we can say that the control system which operates on the time basis is open loop system.

Advantages:

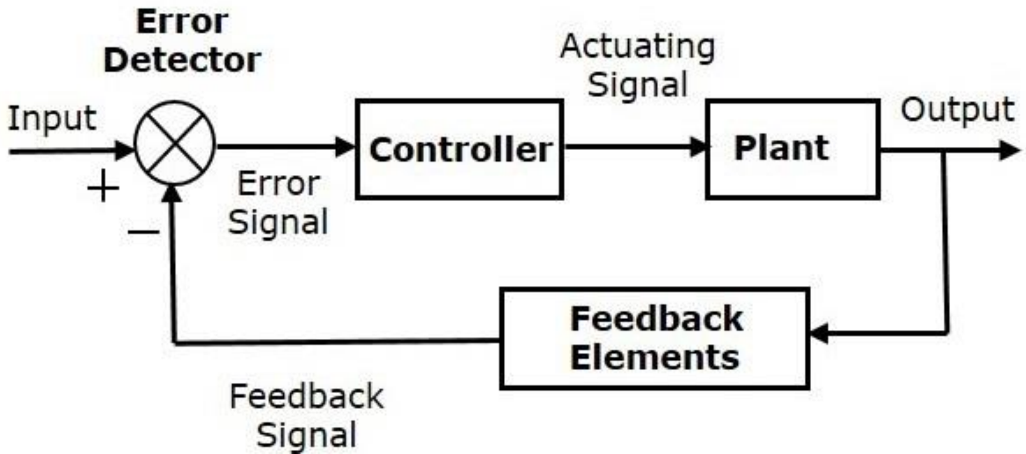
1. Open loop control systems are simple.
2. Open loop control systems are economical.
3. Less maintenance is required and not difficult.
4. Proper calibration is not a problem.

Disadvantages:

1. Open loop systems are inaccurate.
2. These are not reliable.
3. These are slow.
4. Optimization is not possible.

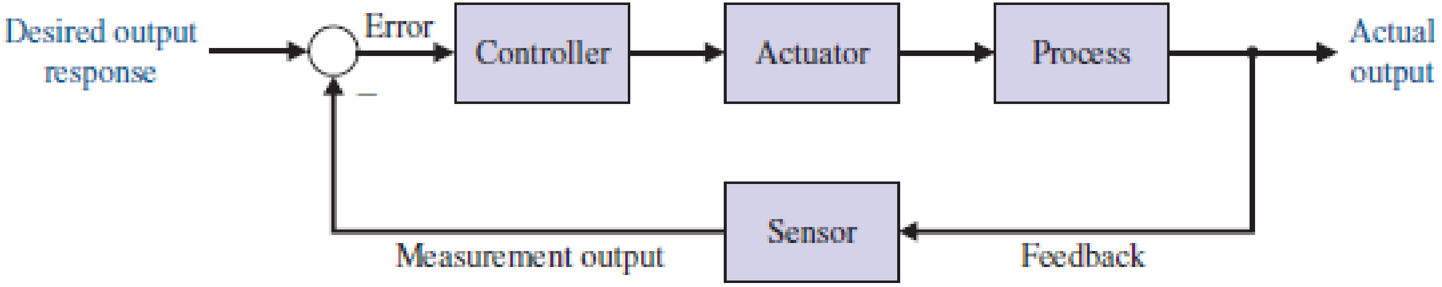
Closed loop control system

- Closed loop control systems are those with feedback. In closed loop control systems the control action is dependent on the desired output. In closed loop control system, the controlled output is measured by the feedback measuring device and compared with the reference input signal, and the error or deviation is minimized by applying manipulated signal by the controller to bring back the output of the system to desired value. The error signal is fed to the controller to reduce the error and desired output is obtained.
- Closed loop control systems are also called feedback control system. is called a **feedback control system or closed loop cont**



Closed-loop feedback control system

FIGURE 1.3 Closed-loop feedback control system (with feedback).



A closed-loop control system utilizes an additional measure of the actual output to compare the actual output with the desired output response. The measure of the output is called the **feedback signal**

A feedback control system is a control system that tends to maintain a prescribed relationship of one system variable to another by comparing functions of these variables and using the difference as a means of control. With an accurate sensor, the measured output is a good approximation of the actual output of the system.

Closed-loop feedback control system

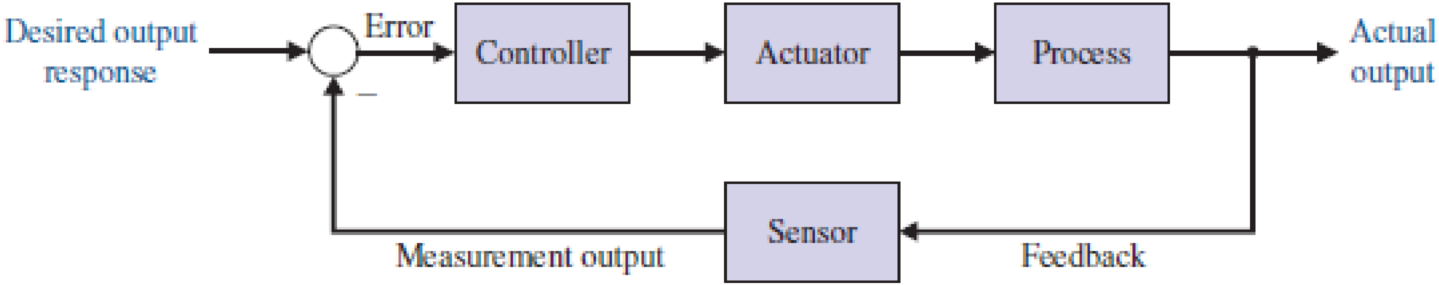


FIGURE 1.3
Closed-loop feedback control system (with feedback).

A feedback control system often uses a function of a prescribed relationship between the output and reference input to control the process, Often the difference between the output of the process under control and the reference input is amplified and used to control the process so that the difference is continually reduced.

In general, the difference between the desired output and the actual output is equal to the error, which is then adjusted by the controller. The output of the controller causes the actuator to modulate the process in order to reduce the error.

A closed-loop control system uses a measurement of the output and feedback of this signal to compare it with the desired output (reference or command)

Negative Feedback Control System

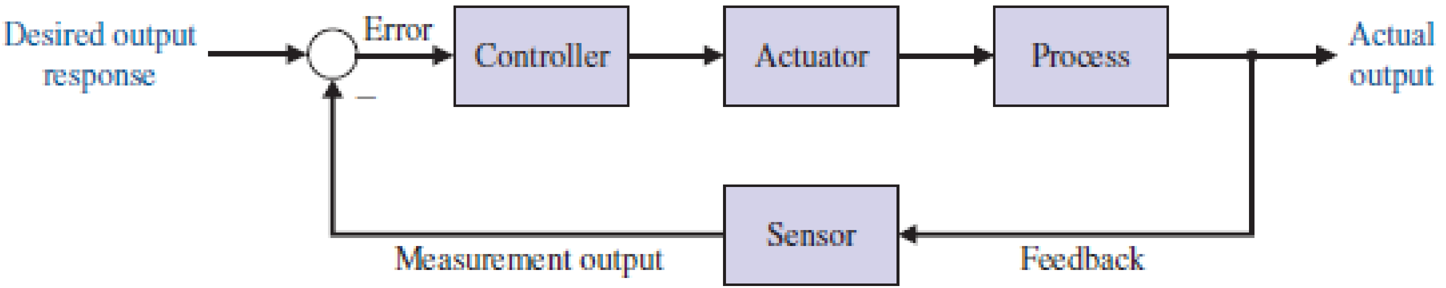


FIGURE 1.3
Closed-loop feedback control system (with feedback).

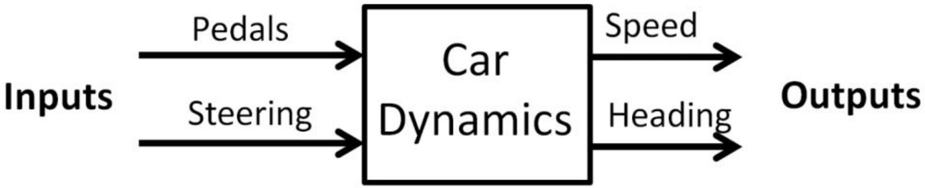
The system shown in Figure is a **negative feedback** control system, because the output is subtracted from the input and the difference is used as the input signal to the controller. **The feedback concept has been the foundation for control system analysis and design**

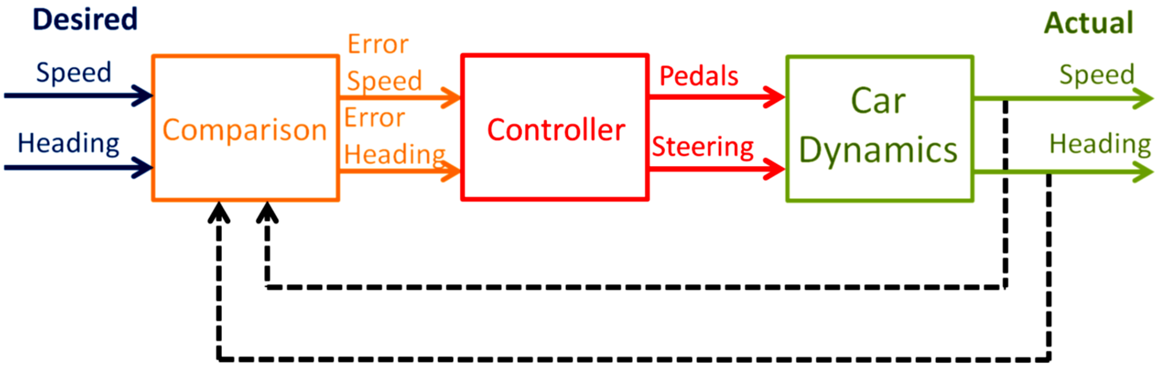
The introduction of feedback enables us to control a desired output and can improve accuracy, but it requires attention to the issue of stability of response

Feedback's inherent capability is that its parameter can be adjusted to alter both transient and steady state responses as together they are referred to as time responses.

An example of a closed-loop control system is a person steering an automobile (assuming his or her eyes are open) by looking at the auto's location on the road and making the appropriate adjustments.

Open Loop Car Block Diagram





The driver controls the car’s pedals (acceleration and braking) and the steering wheel, causing the car to change its speed and heading.

Example: In a room we need to regulate the temperature and humidity for comfortable living. Air- conditioners are provided with thermostat. By measuring the actual room temperature and compared it with desired temperature, an error signal is produced, the thermostat turns ON the compressor or OFF the compressor.

Advantages:

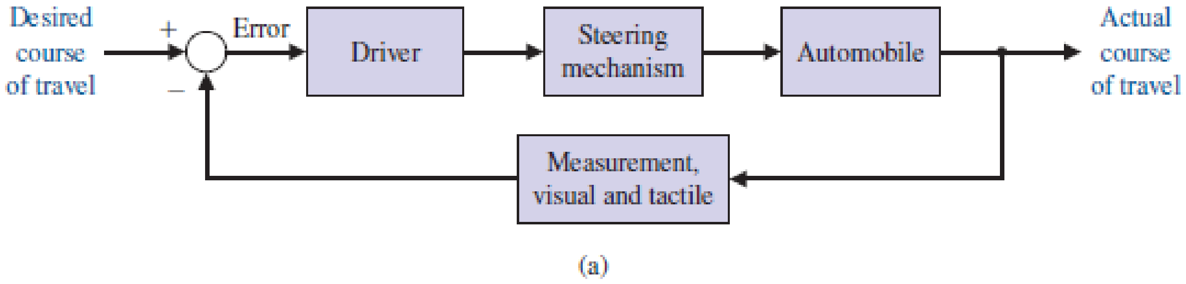
1. These systems are more reliable.
2. Closed loop systems are faster.
3. A number of variables can be handled simultaneously.
4. Optimization is possible.

Disadvantages:

1. Closed loop systems are expensive.
2. Maintenance difficult.
3. Complicated installation.

EXAMPLES OF CONTROL SYSTEMS

EXAMPLE **Automated vehicles**



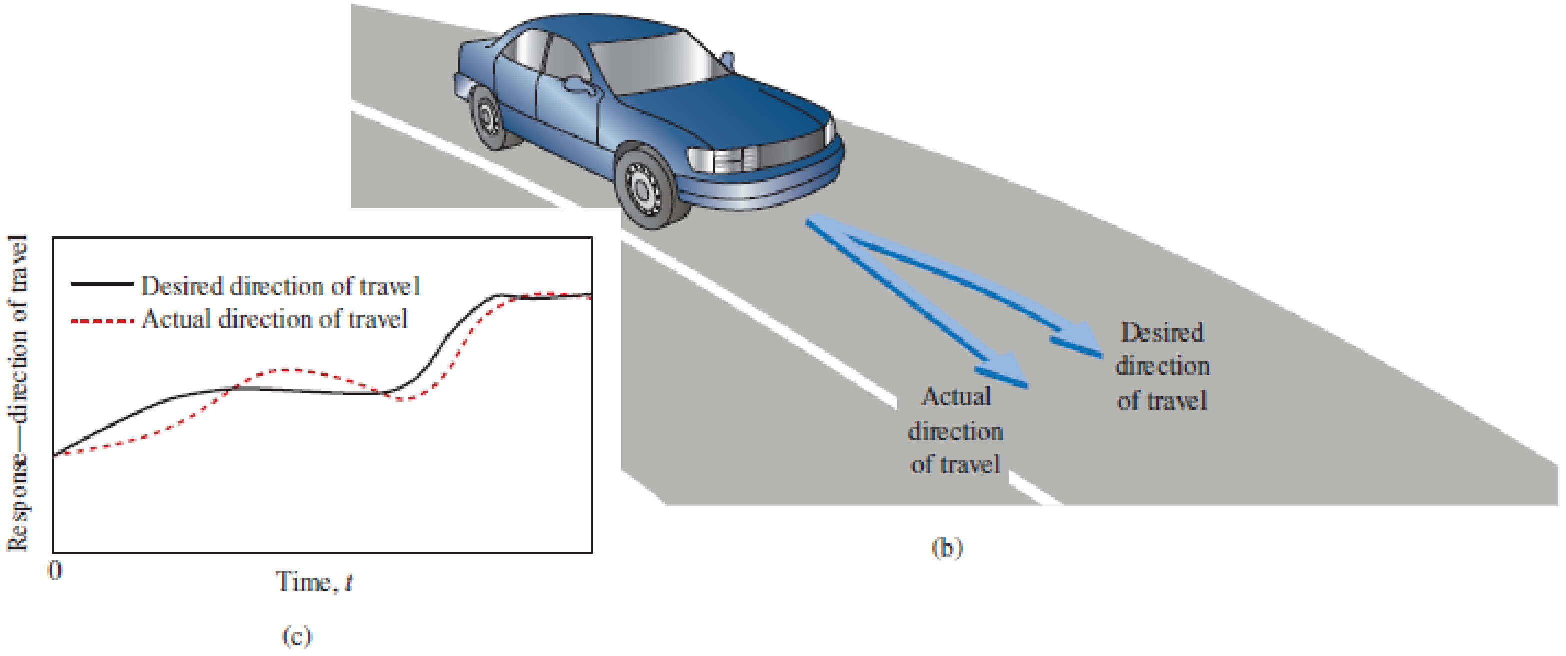


FIGURE 1.10
(a) Automobile steering control system. (b) The driver uses the difference between the actual and the desired direction of travel to generate a controlled adjustment of the steering wheel. (c) Typical direction-of-travel response.

Human-in-the-loop control

A basic, manually controlled closed-loop system for regulating the level of fluid in a tank is shown in Figure 1.11.

The input is a reference level of fluid that the operator is instructed to maintain. (This reference is memorized by the operator.) The power amplifier is the operator, and the sensor is visual. The operator compares the actual level with the desired level and opens or closes the valve (actuator), adjusting the fluid flow out, to maintain the desired level

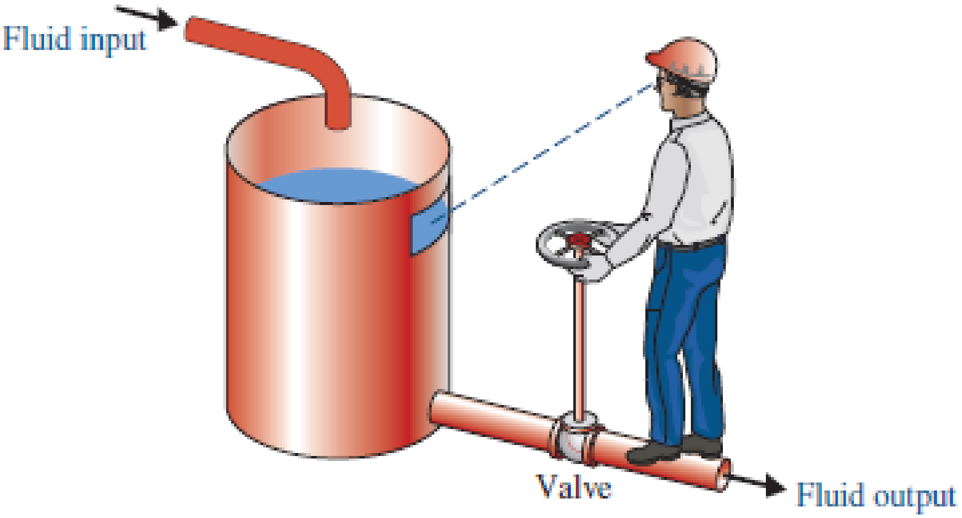


FIGURE 1.11
A manual control system for regulating the level of fluid in a tank by adjusting the output valve. The operator views the level of fluid through a port in the side of the tank.

Another example,

if a ship is heading incorrectly to the right, the rudder is actuated to direct the ship to the left

Comparison between open loop and closed loop system

Open loop control system	Closed loop control system
➢ The control action is independent of the output	➢ The control action is dependent on the output.
➢ There is no feedback element	➢ There is a feedback element.
➢ They are less accurate and accuracy depends on calibration	➢ Increased accuracy, as the actuating error signal is fed to the controller to reduce the error and bring the output of the system to desired value.
➢ They are not troubled with the instability	➢ Stability is a major problem as they may tend to over-correct errors leading to oscillation of output
➢ They are highly sensitive to variation in system characteristics	➢ Reduced sensitivity to variation in the system characteristics.
➢ Presence of non linearities, distortion and any external disturbance may lead to the non-performance of the desired task.	➢ The system is relatively insensitive to non-linearities, distortion and external disturbance.
➢ The open loop control system is cheaper and consumes less power as less number of components are involved	➢ The closed loop control system is higher in cost and consumes more power as more number of components are involved

Comparison between Open loop and Closed loop control systems

Open loop control system	Closed loop control system
1. Accuracy depends on the calibration of the input	1. More accurate due to the presence of feedback
2. Intelligent controlling action not possible	2. Intelligent controlling action
3. Less components required to construct	3. More components required to construct
4. More stable in operation	4. Stability depends on system components
5. Operation affected if non-linearity present	5. Better performance compared to open loop system if non-linearity present
6. Simple to construct	6. Complicated design

Open Loop System	Closed Loop System
1. These are not reliable.	These are reliable.
2. It is easier to build.	It is difficult to build.
3. If calibration is good, they perform accurately.	They are accurate because of feedback.
4. Open loop systems are generally more stable.	These are less stable
5. Optimization is not possible. Closed Loop System	Optimization is possible.

Control System Applications

Liquid level control system

Paper Tension Control System

Antenna Tracking Satellite System

Computer Hard Disk Drive Read/Write System

Robotic Control Systems

Sun Tracker Control System

Liquid level control system

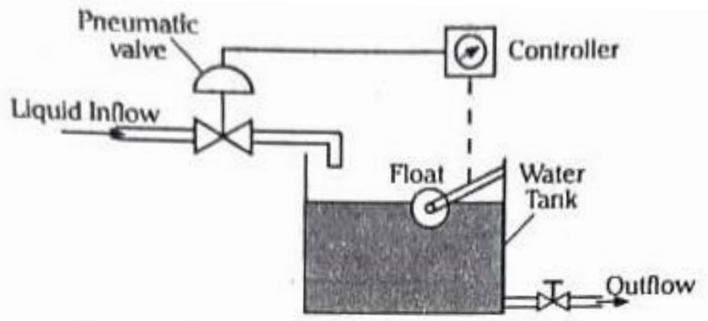


Fig. (a) Liquid level control system

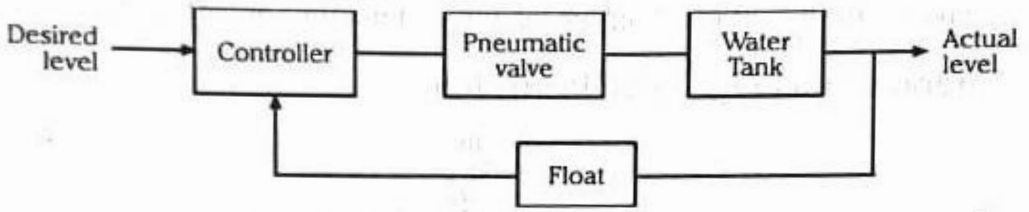


Fig. (b) Block diagram

Shows a schematic arrangement of liquid level control system for a water tank. As the water level increases the ball float on the water moves up which is sensed by controller. Controller inturn controls the opening of the pneumatic valve and closes the valve completely when the desired level of liquid inside the tank is reached. The Block diagram representation is shown

Paper Tension Control System

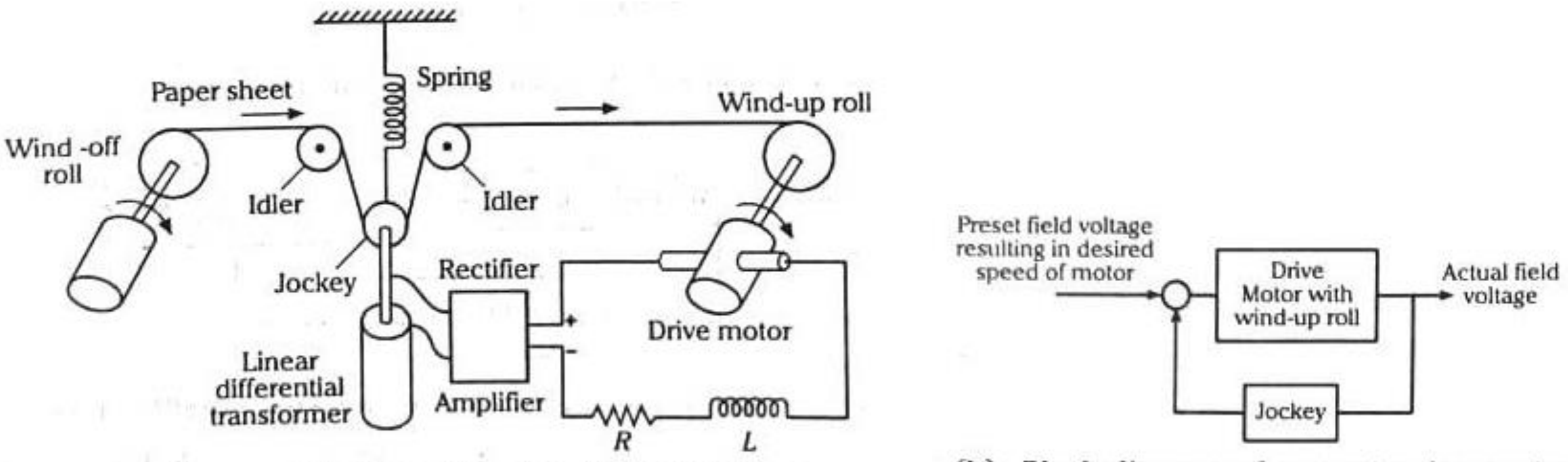


Fig. (a) Paper Tension Control System (b) Block diagram of paper tension control system

In paper processing plant, it is important to maintain a constant tension on the paper between wind off roll and wind up roll. If the reel speed is constant, the diameter of the wind up roll goes on increasing and therefore the tension of the paper also varies. Any increase in the tension of the paper causes tearing of the paper. Therefore the tension control is required by adjusting the take up motor speed

Paper Tension Control System

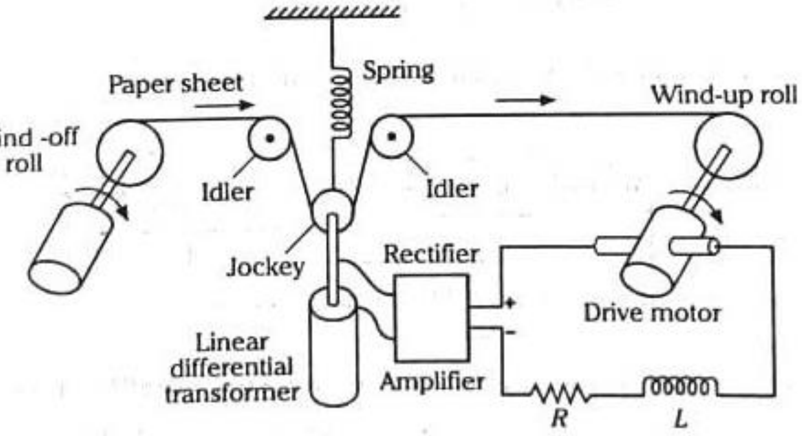
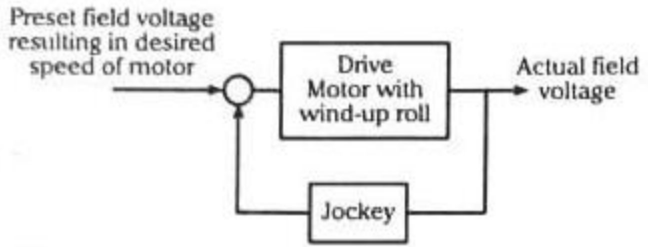


Fig. (a) Paper Tension Control System



(b) Block diagram of paper tension control system

The schematic arrangement of the paper tension control system is shown in Fig. (a) which consists of two idlers and one jockey roller. Whenever the tension of the paper is increased, the jockey roller moves upward and under decreased tension the jockey roller moves down. This vertical motion of the Jockey is used to change the field current of the drive motor and hence the speed of the wind-up roll also gets adjusted to maintain the constant tension of the paper. The block diagram of the system is shown in Fig.(b).

Antenna Tracking Satellite System

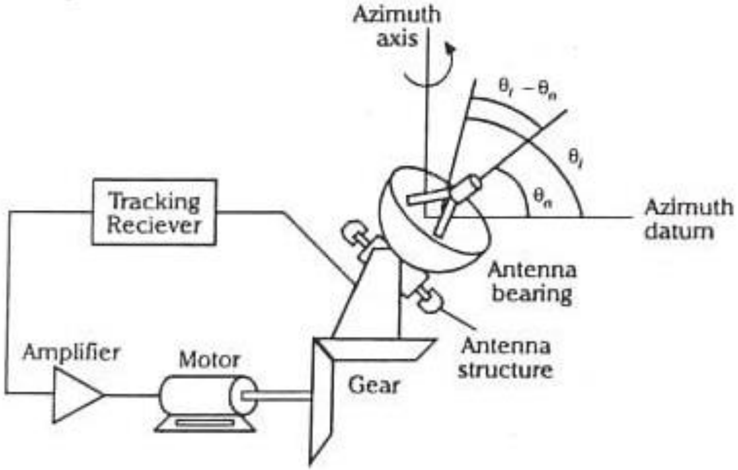


Fig. (a) Antenna tracking satellite system

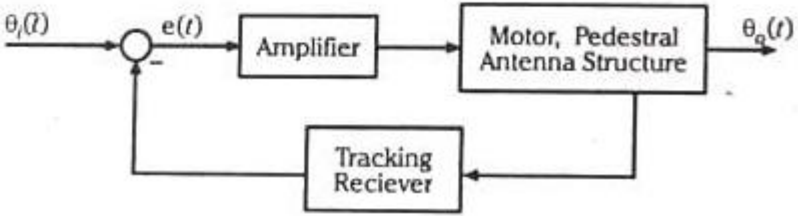


Fig. (b) Block diagram of Antenna tracking satellite system

The schematic arrangement of antenna tracking satellite system is shown is Fig. (a).

The elevation angle and azimuth or horizontal bearing of the antenna must be controlled so that the antenna points at the satellite and receives strongest possible signal from it. The tracking receiver receives the signal and provides a measure of the angular misalignment between the antenna and the satellite.

Antenna Tracking Satellite System

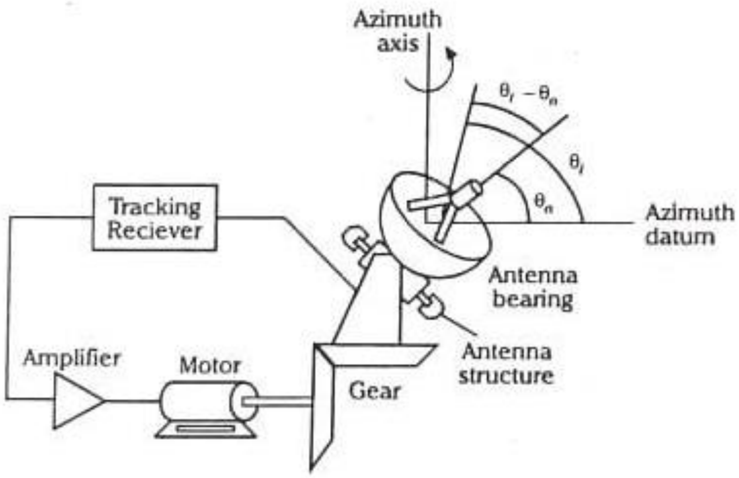


Fig. (a) Antenna tracking satellite system

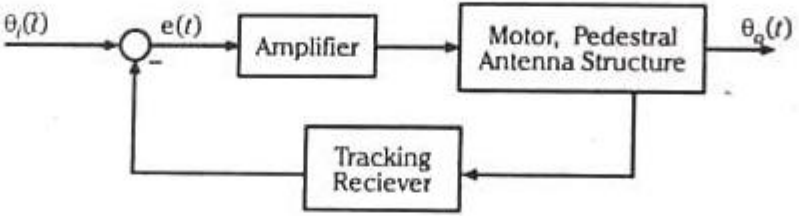


Fig. (b) Block diagram of Antenna tracking satellite system

The azimuth error from the tracking receiver is amplified by the amplifier and the amplified signal drives the drive motor which inturn drives the antenna about its azimuth axis to the desired position and corrects error in misalignment. The block diagram of the system is shown in Fig. (b)

Computer Hard Disk Drive Read/Write System

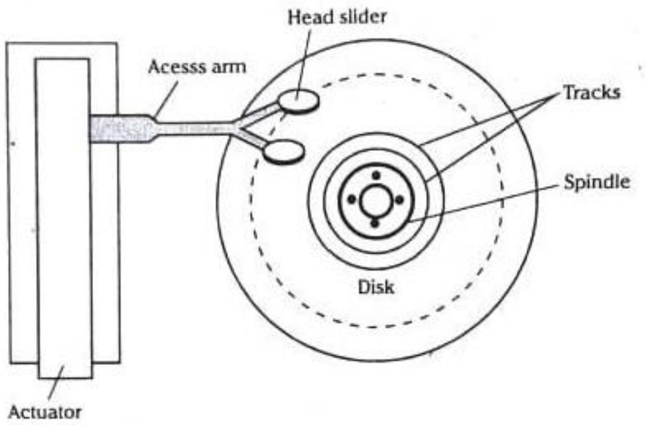


Fig. (a) Schematic arrangement of Hard disk drive system

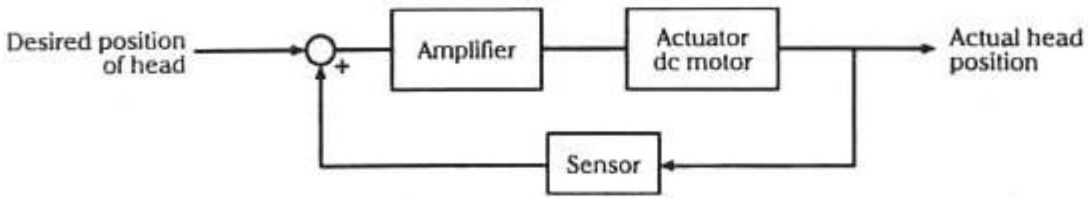


Fig. (b) Block diagram of Computer Hard disk drive system

A hard disk is a circular disk coated with magnetic materials such as metal oxide which acts as storage device. A hard disk is a sealed unit that contains a stack of metal plates called platters. These platters are mounted horizontally on a vertical axle which will rotate at a speed of about 7000 rpm to 10,000 rpm. Magnetic head of the hard disk drive is used to read/write stored data.

Computer Hard Disk Drive Read/Write System

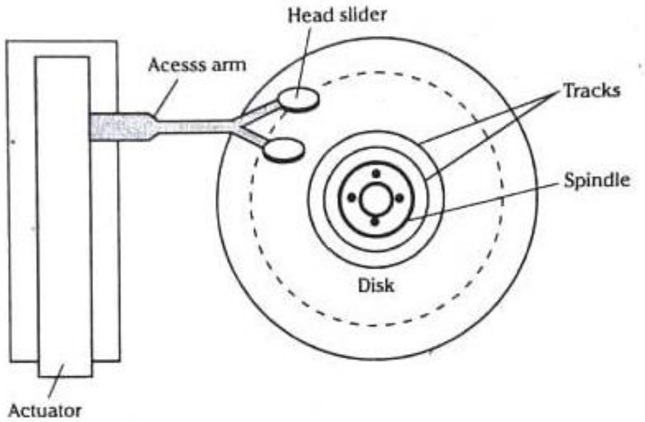


Fig. (a) Schematic arrangement of Hard disk drive system

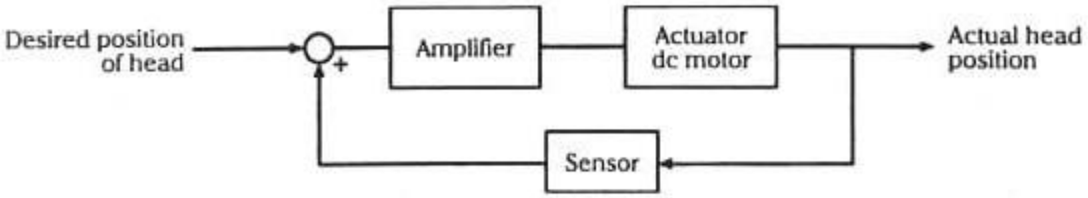
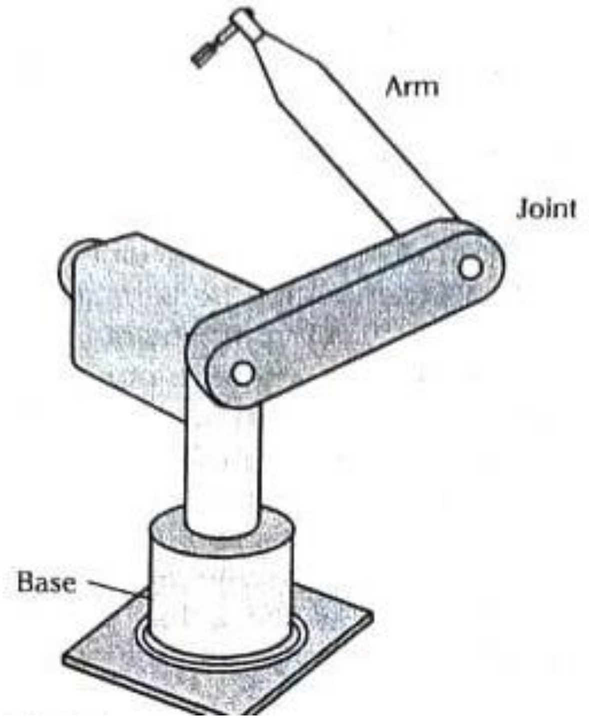


Fig. (b) Block diagram of Computer Hard disk drive system

Each platter will have one magnetic read/write head. The access arm controls the movement of all the heads simultaneously. The read/write head floats just above the disk on a cushion of air. The clearance between the head and disk will be three hundred times less than the thickness of hair. If it is a notebook computer, the disk drives will be subjected to wobble, physical shocks on the spindle bearings. The disk drive must position the access arm and the read/write heads accurately despite the disturbance. The controller takes the difference of the actual and desired position and generates an error. The error is amplified and fed to the drive motor

Robotic Control Systems



Schematic diagram of Robot

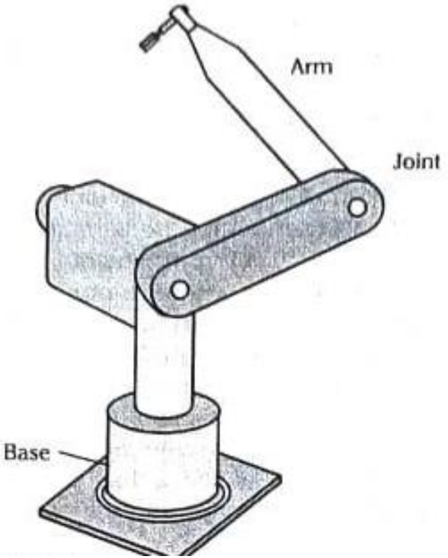
Robotic systems are classified according to the nature of actuators as electric, hydraulic or pneumatic type.

The electric type uses either dc motor or step motor. For medium power requirements dc motors are suitable while for small power requirement step motors are currently used.

The hydraulic type is used in industries where large power is required.
Pneumatic type is suitable for situation where linear motion is involved.

Robotic systems are also classified according to their coordinate system they possess as Rectangular coordinate. Polar coordinate, Circular coordinate and Multiple joint type robot system.

Robotic Control Systems



Schematic diagram of Robot

Fig. shows the schematic diagram of Multiple joint type Robot arm system, The Robots are interfaced to a digital controllers. The Robot arm system will have force sensing device which will be in the form of semiconductor strain gauge. Slip sensing device such as roller device to the contact surface. Here the rotation angle of the roller gives the amount of slip. The tasks performed by human operator will be stored in magnetic disks. Then robot will repeat this sequence of operation faithfully and carry out desired tasks. The robots can perform the task at a greater speed with higher precision.

Sun Tracker Control System

Solar energy can be converted into other forms of energy. In these energy convertors Sun tracker system is an important device in which the controller constantly calculates the Sun's rate for the azimuth axes and elevation of control. The controller uses the sun rate and sun sensor information and compares with the input and the error is amplified by the amplifier and generate proper motor command to slew the telescope mount

Requirements of a good control system

- **Accuracy**
- **Sensitivity**
- **External Disturbance or Noise**
- **Stability**
- **Bandwidth**
- **Speed**
- **Oscillations**

Multi Loop Feedback system

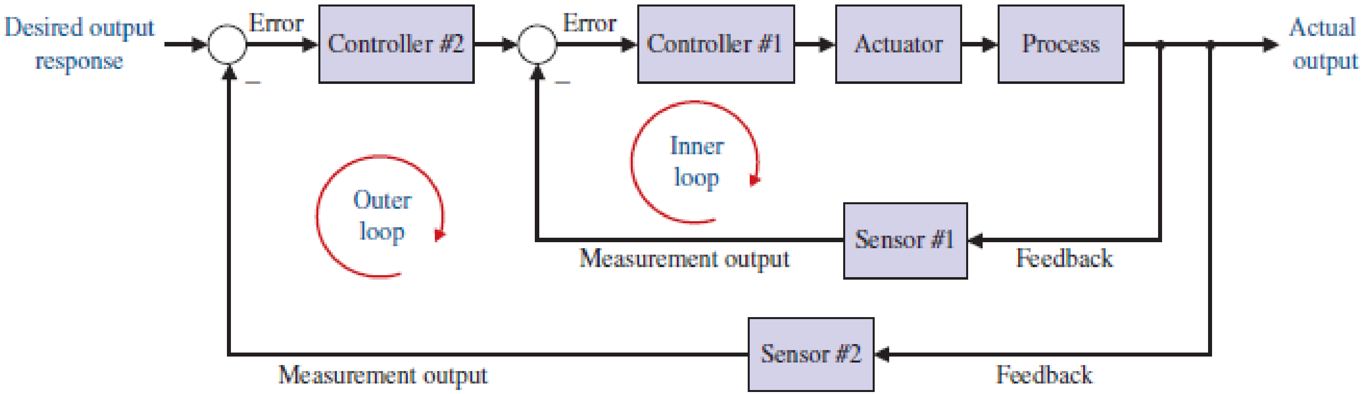


FIGURE 1.5 Multiloop feedback system with an inner loop and an outer loop.

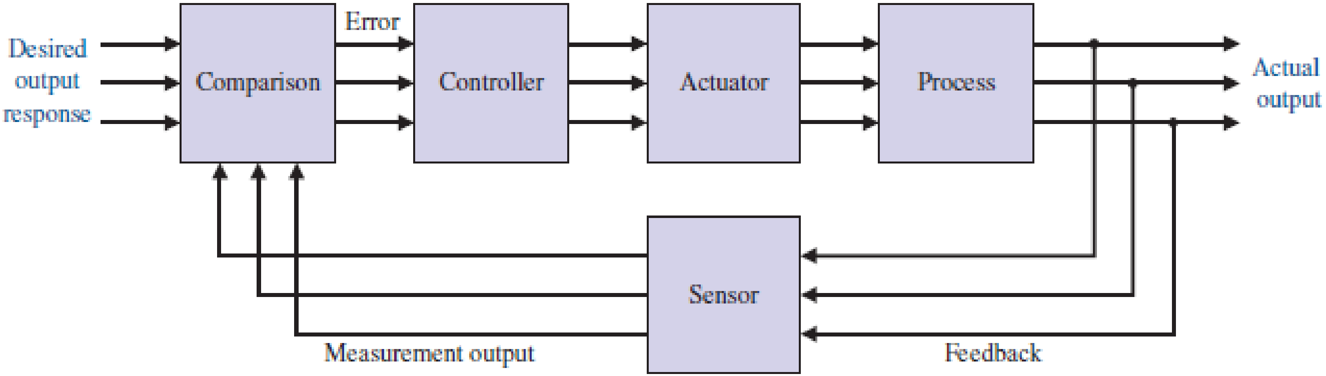


FIGURE 1.6 Multivariable control system.

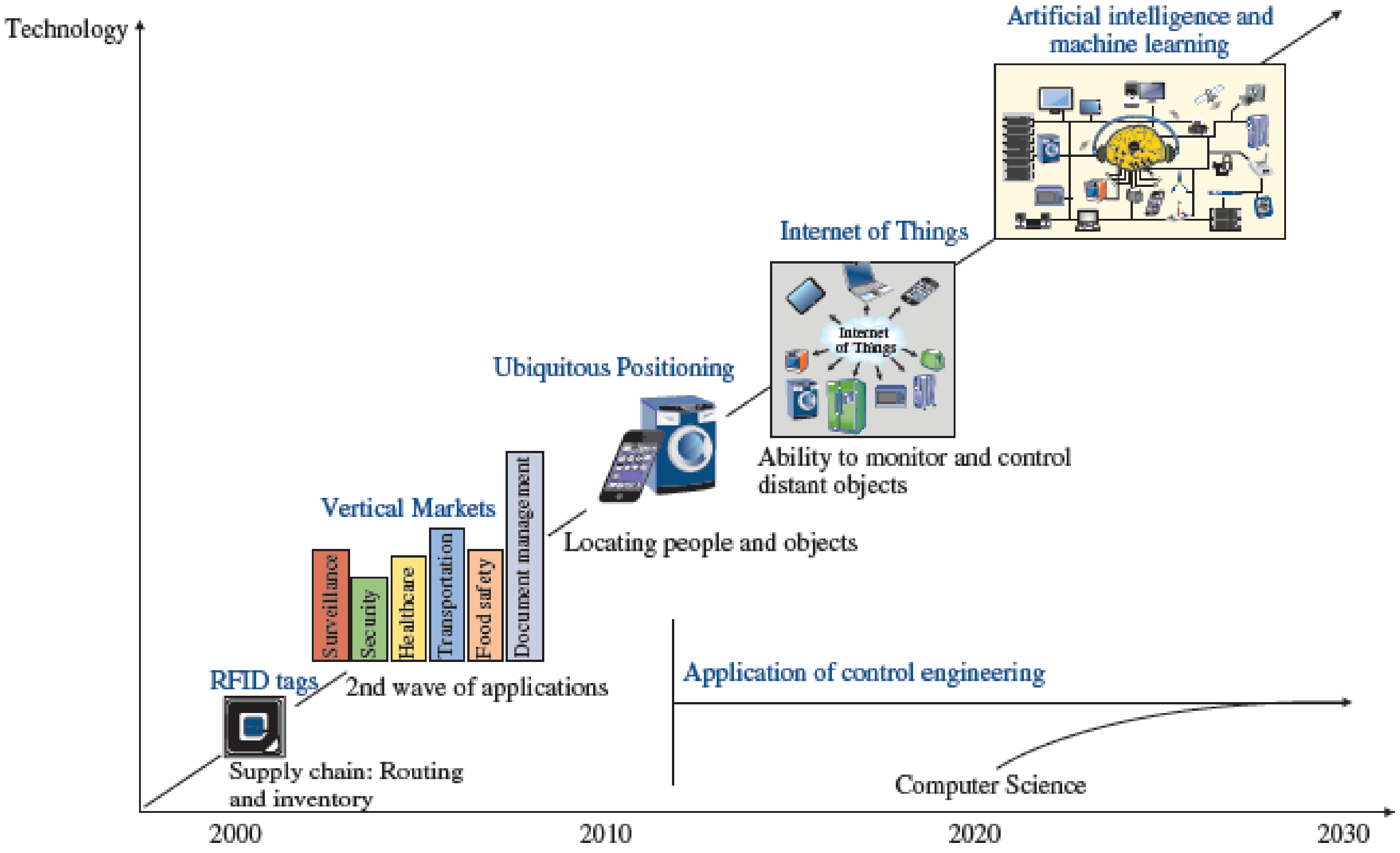


FIGURE 1.9 Technology roadmap to the Internet of Things enhanced with artificial intelligence with applications to control engineering (Source: SRI Business Intelligence).

Definitions In Control System

System : A system is a collection or combination of components that are related to one another or acting together to perform certain objective. A system is not restricted to physical ones, but also be extended to dynamic phenomena such as economics.

Control : Control means 'to regulate, direct or command. In control system control means measuring the output variable and comparing it with input variable and correct any. deviation of the measured value from a desired value by applying suitable manipulated variable.

Control system : A control system is an interconnection of components which is used to command, direct, or regulate itself or any other system

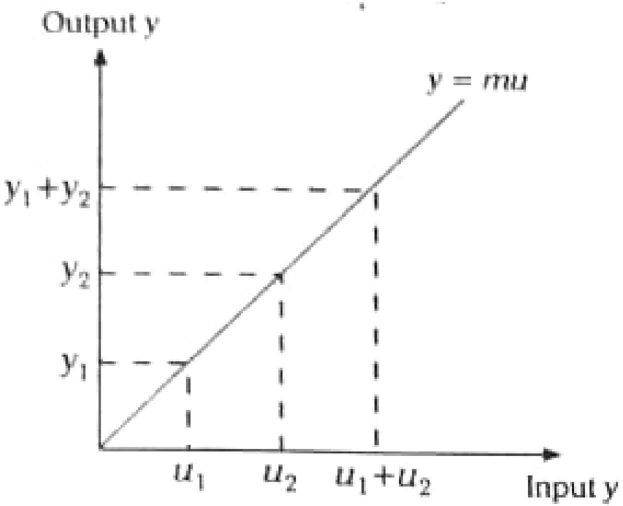
Single Input - Single Output (SISO) and Multiple Input - Multiple output (MIMO) system:

SISO system will hare only one input and one output. Servo mechanism is an example of SISO system. One command Input that is the position of the arm of input potentiometer and one controlled output.

MIMO System will have more than one Input as well as outputs. Steam generation Boiler is an example of MIMO system which involves multiple inputs- Pressure & temperature input and multiple output-pressure and temperature outputs.

Linear and Non-linear control systems

- In practice all linear systems are non linear to some degree.
- **The linear systems** are represented by linear differential equation with constant coefficients
- The principle of superposition can be applicable to linear control systems. According to the principle of superposition, if two separate inputs U1, and U2 are applied to a system which gives rise to outputs y1, and y2, respectively, then the response produced by the simultaneous application of the inputs (ie., when these inputs are super imposed) is equal to the sum of the two individual outputs (Y1 + Y2)



Mathematical Model of Physical Systems

To analyse and design a control system the first step is to formulate quantitative description of the system. This process of obtaining desired description is called modelling. A physical system can be modelled in a number of ways. For example a satellite can be modelled as a point, a flexible body or a rigid body depending upon the type of study that is to be carried out. Thus the physical models are developed with assumption

The next step is to obtain mathematical model by applying physical laws to these physical models. Generally control systems may be modelled as a scalar differential equations or vector matrix notations.

Mathematical Model of Physical Systems

Such equations provide complete description of the system and for any given input or stimulus, the response is obtained by solving these equation. However this method can be cumbersome and difficult for the designer to handle.

Instead of solving differential equation, the otherway is to convert differential equation to algebraic equation. We then rearrange the algebraic equation to obtain transfer function form and use frequency response methods. Then the simulations are done in the computer to predict whether the systems works satisfactorily or not.

Mathematical models of dynamic processes are often derived using physical laws such as Newton's and Kirchhoff's

Modelling of Mechanical System

- In mechanical systems, the motion can be translational, rotational or combination of these two.
- If the motion takes place along a straight line, it is called translational motion. The displacement(x), velocity and acceleration are the variables which are used to describe the translational motion.
- The rotational motion involves motion about the fixed axis. The angular displacement, angular velocity and angular accelerations are the variable which are used to describe the rotational motion.
- In both the cases Newtons law of Motion is applied to obtain mathematical model.
- Mechanical system that are generally modelled by means of three elements are mass, linear spring and friction in translational and rotational system.

Modelling of Mechanical System

- The equations governing the motion of mechanical system are formulated from Newton's law of motion.

- Newton's law of motion states that the algebraic sum of forces acting on a rigid body in the given direction is equal to the product of the mass of the body and its acceleration in the same direction.

Mathematical Model of Mechanical Systems

- **Mass**: Mass is considered as a property of an element that stores the kinetic energy of translation motion. If W denotes the weight of a body, the mass M is given by **M=W/g**, where g is the acceleration of free fall of the body due to gravity. One end of Mass is aways connected to ground

- **Linear Spring:** in Practice, a liner spring may be model of a natural spring or a compliance of a cable or belt, Spring is considered to be an element that stores potential energy. All spring in real life are nonlinear to some extent.

- **Dashpot or viscous friction units:** Whenever there is a motion or tendency of motion between two physical elements, frictional forces exists.

The graphical and symbolic notations for all three

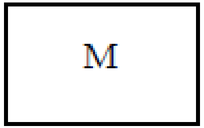


Fig 1-8 a) Mass

Fig 1-8 b) Spring

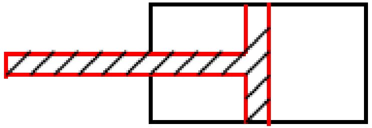


Fig 1-8 c) Dashpot

Modeling of Translational Mechanical Systems

- ➢Translational mechanical systems move along a **straight line**.

- ➢These systems mainly consist of three basic elements, Those are **Mass, Spring and Friction (**dashpot or damper).

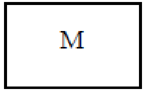


Fig 1-8 a) Mass

Fig 1-8 b) Spring

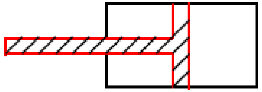


Fig 1-8 c) Dashpot

- ➢ These three elements represents three essential phenomena which occur in various ways in mechanical systems.
- ➢ The weight of the mechanical system is represented by the element mass and it is assumed to be concentrated at the center of the body.
- ➢ The elastic deformation of the body can be represented by a spring.
- ➢ The friction existing in Translational /rotating mechanical system can be represented by the dash-pot. The dashpot is a piston moving inside a cylinder filled with viscous fluid.

Modeling of Translational Mechanical Systems

- When a force is applied to a translational mechanical system, it is opposed by opposing forces due to mass, friction and elasticity of the system. The force acting on a mechanical body are governed by Newton's second law of motion. For translational systems it states that the sum of forces acting on a body is zero.
- (or Newton's second law states that the sum of applied forces is equal to the sum of opposing forces on a body).

LIST OF SYMBOLS USED IN MECHANICAL TRANSLATIONAL SYSTEM

x = Displacement, m

$v = \frac{dx}{dt}$ = Velocity, m/sec

$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$ = Acceleration, m/sec^2

f = Applied force, N (Newtons)

f_m = Opposing force offered by mass of the body, N

f_k = Opposing force offered by the elasticity of the body (spring), N

f_b = Opposing force offered by the friction of the body (dash - pot), N

M = Mass, kg

K = Stiffness of spring, N/m

B = Viscous friction co-efficient, N-sec/m

Note : Lower case letters are functions of time

Force Balance Equation of idealized elements

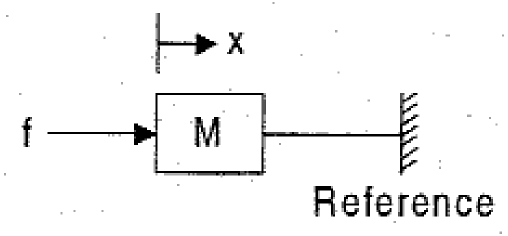
Let us now see the force opposed by these three elements individually

Mass

Consider an ideal mass element shown in fig. which has negligible friction and elasticity.

Let a force be applied on it. The mass will offer an opposing force which is proportional to acceleration of the body

Ideal Mass element



Where,

- **F** is the applied force
- **F**$_m$ is the opposing force due to mass
- **M** is mass
- **a** is acceleration
- **x** is displacement

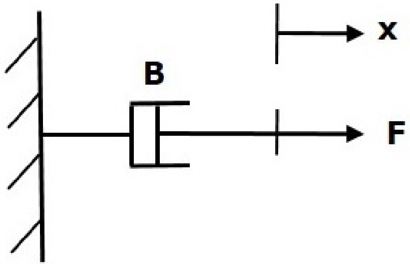
$$F_m \propto a$$

$$\Rightarrow F_m = Ma = M\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

By Newton's second law $F = F_m = M\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$

Dashpot

Consider an ideal frictional element dashpot shown in fig. which has negligible mass and elasticity.
Let a force be applied on it. The dash-pot will offer an opposing force which is proportional to velocity of the body



Reference

Ideal dashpot with one end fixed to reference

Where,

$$F_b \propto v$$

- F_b is the opposing force due to friction of dashpot
- B is the frictional coefficient
- v is velocity
- x is displacement

$$\Rightarrow F_b = Bv = B\frac{dx}{dt}$$

By Newton's second law $F = F_b = B\frac{dx}{dt}$

When the dashpot has displacement at both ends as shown fig, the opposing force is proportional to differential velocity or Velocity difference,.

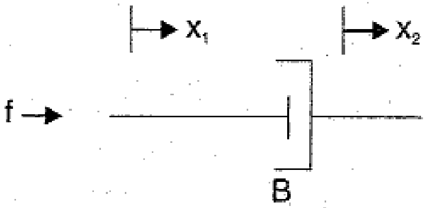


Fig : Ideal dashpot with displacement at both ends.

$$f_b \propto \frac{d}{dt}(x_1 - x_2) \quad \text{or} \quad f_b = B\frac{d}{dt}(x_1 - x_2)$$

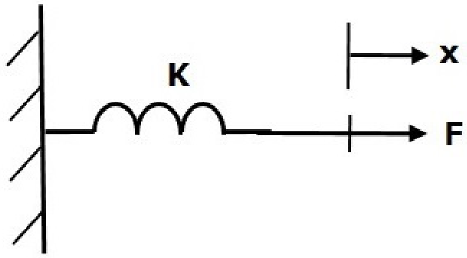
By Newton's second law $\therefore \boxed{f = f_b = B\frac{d}{dt}(x_1 - x_2)}$

Spring

Consider an ideal elastic element spring shown in fig, which has negligible mass and friction.

Let a force be applied on it. The spring will offer an opposing force which is proportional to displacement of the body

Reference



Ideal Spring with one end fixed to reference

Where,

- **F** is the applied force
- $\mathbf{F_k}$ is the opposing force due to elasticity of spring
- **K** is spring constant
- **x** is displacement

$$F \propto x$$

$$\Rightarrow F_k = Kx$$

By Newton's second law $F = F_k = Kx$

When the spring has displacement at both ends as shown in fig. the opposing force is proportional to differential displacement or Displacement difference.

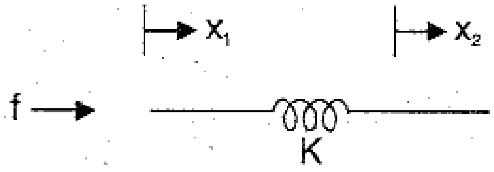
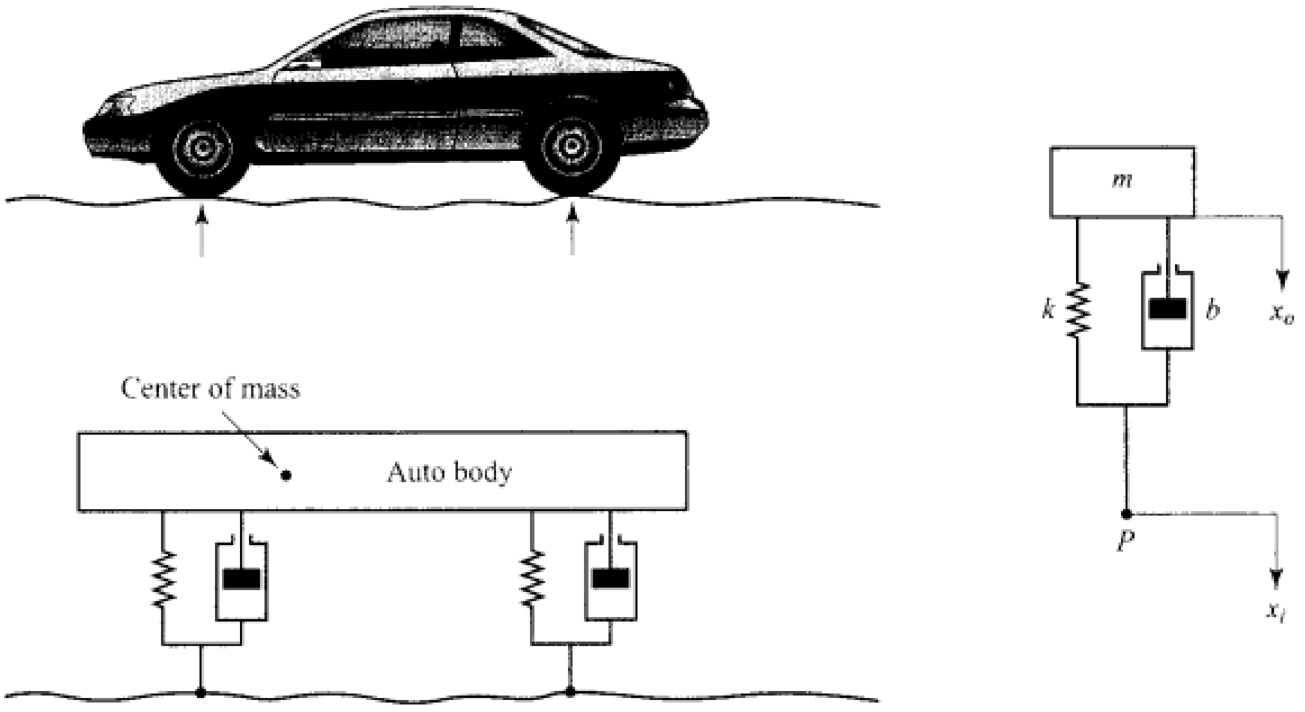


***Fig** : Ideal spring with displacement at both ends.*

$f_k \propto (x_1 - x_2)$ or $f_k = K(x_1 - x_2)$

By Newton's second law $\therefore \boxed{f = f_k = K(x_1 - x_2)}$ ....

A SCHEMATIC DIAGRAM OF AN AUTOMOBILE SUSPENSION SYSTEM



(a) (b)

(a) Automobile suspension system; (b) simplified suspension system.

Guideline to determine the Transfer Function of Mechanical Translational System

1. In mechanical translational system, the differential equations governing the system are obtained by writing force balance equations at nodes in the system. The nodes are meeting point of elements. Generally the nodes are mass elements in the system. In some cases the nodes may be without mass element.

2. The linear displacement of the masses (nodes) are assumed as X1, X2, X3, etc., and assign a displacement to each mass(node). The first derivative of the displacement is velocity and the second derivative of the displacement is acceleration

Guideline to determine the Transfer Function of Mechanical Translational System

3. Draw the free body diagrams of the system.

   The free body diagram is obtained by drawing each mass separately and then marking all the forces acting on that mass (node). Always the opposing force acts in a direction opposite to applied force. The mass has to move in the direction of the applied force. Hence the displacement, velocity and acceleration of the mass will be in the direction of the applied force. If there is no applied force then the displacement, velocity and acceleration of the mass will be in a direction opposite to that of opposing force.

4. For each free body diagram, write one differential equation by equating the sum of applied forces to the sum of opposing forces.

Guideline to determine the Transfer Function of Mechanical Translational System

5. Take Laplace transform of differential equations to convert them to algebraic equations. Then rearrange the s-domain equations to eliminate the unwanted variables and obtain the ratio between output variable and input variable. This ratio is the transfer function of the system

Note : Laplace transform of $x(t) = \mathcal{L}\{x(t)\} = X(s)$

$$\text{Laplace transform of } \frac{dx(t)}{dt} = \mathcal{L}\left\{\frac{d}{dt}x(t)\right\} = s\,X(s) \quad \text{(with zero initial conditions)}$$

$$\text{Laplace transform of } \frac{d^2x(t)}{dt^2} = \mathcal{L}\left\{\frac{d^2}{dt^2}x(t)\right\} = s^2\,X(s) \quad \text{(with zero initial conditions)}$$

The free-body diagram is obtained

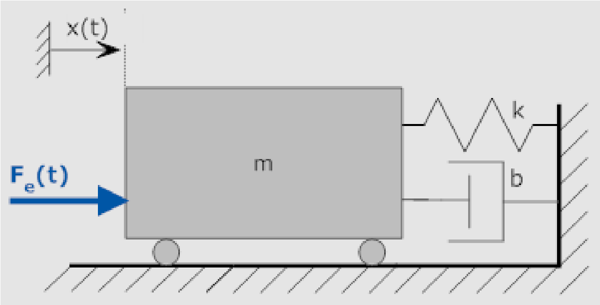
The free-body diagram is created by drawing each mass separately and marking all the forces acting on the node. Write one differential equation for each of the free body diagram and take the Laplace transform and convert the differential equations into the algebraic equations. Rearrange the equations in the s-domain to find the output and input ratio by eliminating the unwanted variables. In this way, we can easily calculate the transfer function of the free body diagram.

1. Drawing each mass separately.
2. By marking all the forces acting on the node
3. Write one differential equation for each of the free body diagram
4. Take Laplace transform of differential equations and rearrange the equations in the s-domain.
5. Find the transfer function as the ratio between output and input variable.

Example: Simple Mass-Spring-Dashpot system

Consider a simple system with a mass that is separated from a wall by a spring and a dashpot.

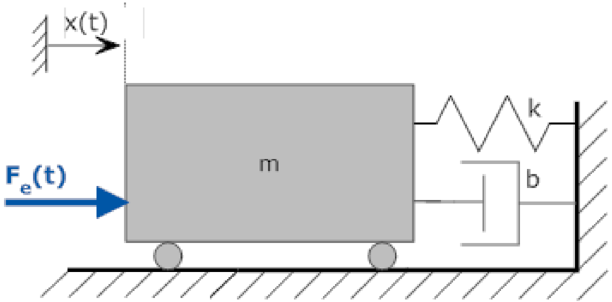
The mass could represent a car, with the spring and dashpot representing the car's bumper. An external force is also shown. Only horizontal motion and forces are considered.

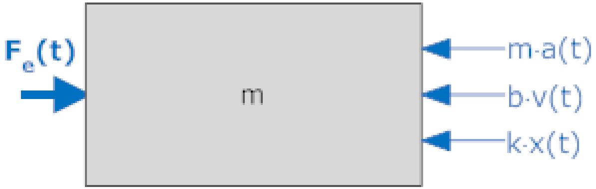


To develop a free body diagram, sum all the forces to zero.

$$\sum_{\text{all}} F = 0$$

To develop a free body diagram, sum all the forces to zero.





$$\sum_{\text{all}} F = 0$$

$$Fe - fm - fb - fk = 0$$

$$F_e(t) - ma(t) - bv(t) - kx(t) = 0$$

$$m\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + kx(t) = F_e(t)$$

There are four forces:

1. An external force (Fe)

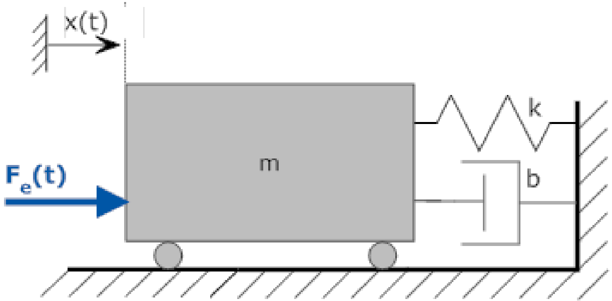
2. A force from the spring. To determine the direction consider that the position "x" is defined positive to the right. If the mass moves in the positive "x" direction, the spring is compressed and exerts a force on the mass. So there will be a force from the spring, $f_k = k \cdot x$, to the left

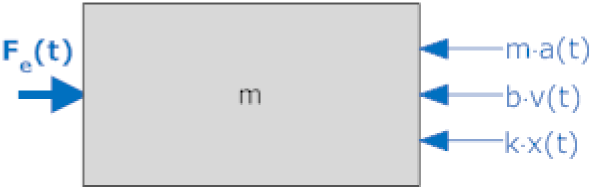
.

3. A force from the Friction/dashpot. By an argument similar to that for the spring there will be a force from the dashpot, $f_b = B \cdot v$, to the left. (The velocity v, is the derivative of displacement x with respect to time.)

4.Finally, there is the inertial force which is defined to be opposed to the defined direction of motion. This is represented by $f_m = M \cdot a$ to the left. (The acceleration a, is the second derivative of displacement x with respect to time.) Don't forget this force

To develop a free body diagram, sum all the forces to zero.





$$\sum_{all} F = 0$$

$$Fe - fm - fb - fk = 0$$

$$F_e(t) - ma(t) - bv(t) - kx(t) = 0$$

$$m\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + kx(t) = F_e(t)$$

This equation is in our standard form that has system outputs (the unknown variables) on the left hand side and system inputs (the known variables) on the right hand side. this is sometimes called input-output notation.

We will often use "dot" notation, using one dot above a variable to denote differentiation:

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = F_e(t)$$

In addition, we will also make it implicit that certain variables are functions of time and omit the "(t)" in equations. If we do so, the equations above become:

$$\sum_{all} F = 0 \qquad Fe - fm - fb - fk = 0$$

$$F_e - ma - bv - kx = 0$$

$$ma + bv + kx = F_e$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_e$$

$$m\ddot{x} + b\dot{x} + kx = F_e$$

This is read as "m x double dot plus b x dot plus k x equals F sub e."

Write the differential equations governing the mechanical system shown in fig 1. and determine the transfer function

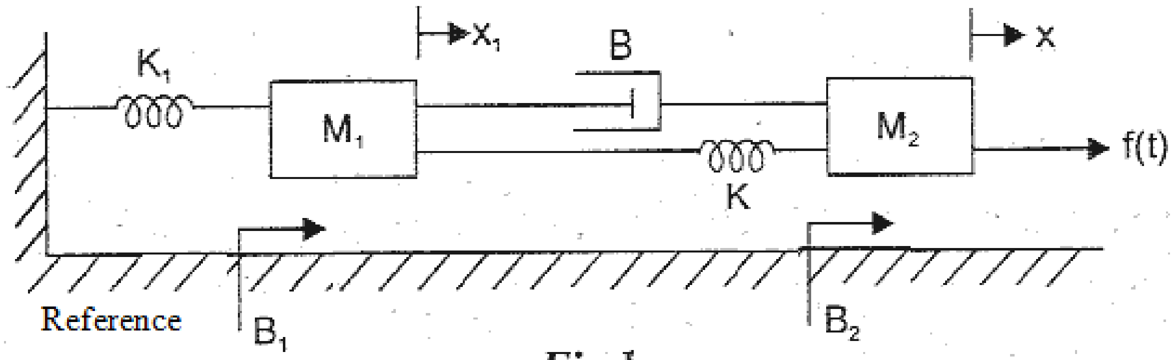


Fig 1.

In the given system, applied force f(t) is the input and displacement x is the output.

Let, Laplace transform of $f(t) = \mathcal{L}\{f(t)\} = F(s)$

Laplace transform of $x = \mathcal{L}\{x\} = X(s)$

Laplace transform of $x_1 = \mathcal{L}\{x_1\} = X_1(s)$

Hence the required transfer function is $\dfrac{X(s)}{F(s)}$

The system has two nodes and they are mass M1 and M2. The differential equations governing the system are given by force balance equations at these nodes.

Let the displacement of mass M_1 be X_1. The free body diagram of mass M_1 is shown in fig 2. The opposing forces acting on mass M_1 are marked as f_{m1}, f_{b1}, f_b, f_{k1}, and f_k

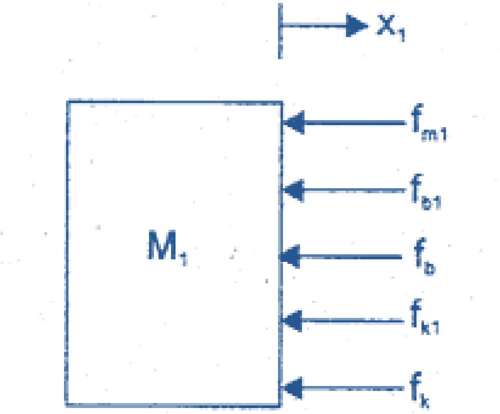


Fig 2 : Free body diagram of mass M_1 (node 1).

$$f_{m1} = M_1 \frac{d^2x_1}{dt^2}\ ; \qquad f_{b1} = B_1 \frac{dx_1}{dt}\ ; \qquad f_{k1} = K_1 x_1;$$

$$f_b = B\frac{d}{dt}(x_1 - x)\ ; \quad f_k = K(x_1 - x)$$

By Newton's second law,

$$f_{m1} + f_{b1} + f_b + f_{k1} + f_k = 0$$

$$\therefore M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B\frac{d}{dt}(x_1 - x) + K_1 x_1 + K(x_1 - x) = 0$$

Laplace transform is used to transfer differential equations to algebraic equations

Laplace transform is used to transfer differential equations to algebraic equations

$$\therefore M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B\frac{d}{dt}(x_1 - x) + K_1x_1 + K(x_1 - x) = 0$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$M_1s^2X_1(s) + B_1sX_1(s) + Bs\,[X_1(s) - X(s)] + K_1X_1(s) + K\,[X_1(s) - X(s)] = 0$$

$$X_1(s)\,[M_1s^2 + (B_1 + B)s + (K_1 + K)] - X(s)\,[Bs + K] = 0$$

$$X_1(s)\,[M_1s^2 + (B_1 + B)s + (K_1 + K)] = X(s)\,[Bs + K]$$

$$\therefore X_1(s) = X(s)\,\frac{Bs + K}{M_1s^2 + (B_1 + B)\,s + (K_1 + K)} \qquad \text{......(1)}$$

The free body diagram of mass M_2 is shown in fig 3. The opposing forces acting on mass M_2 are marked as f_{m2}, f_{b2}, f_b and f_k

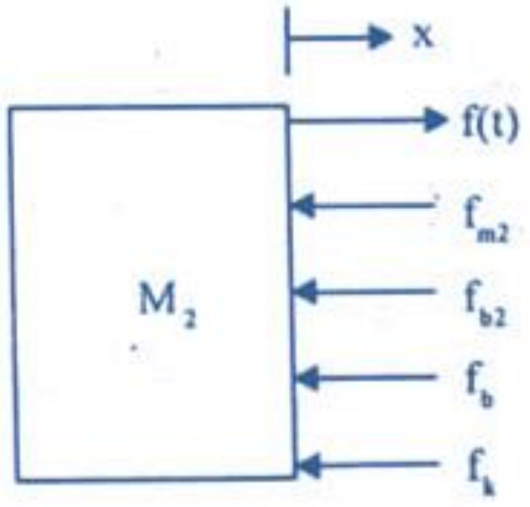


Fig 3 : Free body diagram of mass M_2 (node 2)

$$f_{m2} = M_2 \frac{d^2x}{dt^2} \quad ; \quad f_{b2} = B_2 \frac{dx}{dt}$$

$$f_b = B\frac{d}{dt}(x - x_1) \quad ; \quad f_k = K(x - x_1)$$

By Newton's second law,

$$f_{m2} + f_{b2} + f_b + f_k = f(t)$$

$$M_2 \frac{d^2x}{dt^2} + B_2 \frac{dx}{dt} + B\frac{d}{dt}(x - x_1) + K(x - x_1) = f(t)$$

Laplace transform is used to transfer differential equations to algebraic equations

$$M_2 \frac{d^2x}{dt^2} + B_2 \frac{dx}{dt} + B\frac{d}{dt}(x - x_1) + K(x - x_1) = f(t)$$

On taking Laplace transform of above equation with zero initial conditions we get,

$$M_2 s^2 X(s) + B_2 s X(s) + Bs[X(s) - X_1(s)] + K[X(s) - X_1(s)] = F(s)$$

$$X(s)\,[M_2 s^2 + (B_2 + B)s + K] - X_1(s)[Bs + K] = F(s) \quad .....(2)$$

Substituting for X1(s) from equation (1) in equation (2) we get

$$X(s)\,[M_2s^2+(B_2+B)s+K]-X(s)\,\frac{(Bs+K)^2}{M_1s^2+(B_1+B)s+(K_1+K)}=F(s)$$

$$X(s)\left[\frac{[M_2s^2+(B_2+B)s+K]\,[M_1s^2+(B_1+B)s+(K_1+K)]-(Bs+K)^2}{M_1s^2+(B_1+B)\,s+(K_1+K)}\right]=F(s)$$

$$\therefore \frac{X(s)}{F(s)}=\frac{M_1s^2+(B_1+B)s+(K_1+K)}{[M_1s^2+(B_1+B)s+(K_1+K)]\,[M_2s^2+(B_2+B)s+K]-(Bs+K)^2}$$

The transfer function G(s) of this control system. The definition of the transfer function of a control system is its outputs divided its inputs. In this case, X(s) is the output, F(s) is the input, so we can get G(s) as follows:

$$\therefore \frac{X(s)}{F(s)}=\frac{M_1s^2+(B_1+B)s+(K_1+K)}{[M_1s^2+(B_1+B)s+(K_1+K)]\,[M_2s^2+(B_2+B)s+K]-(Bs+K)^2}$$

Determine the transfer function Y2(s)/F(s) of the system shown in fig

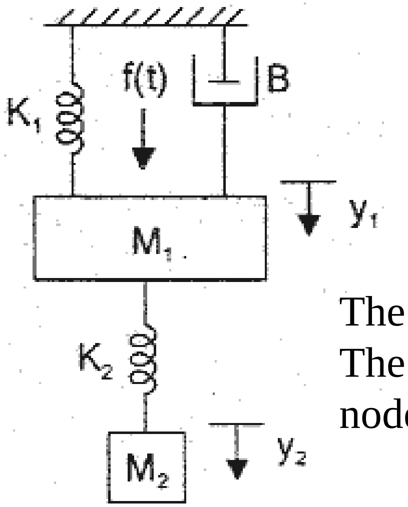


Fig 1.

Let, Laplace transform of $f(t) = \mathcal{L}\{f(t)\} = F(s)$

Laplace transform of $y_1 = \mathcal{L}\{y_1\} = Y_1(s)$

Laplace transform of $y_2 = \mathcal{L}\{y_2\} = Y_2(s)$

The system has two nodes and they are mass M1 and M2.
The differential equations governing the system are the force balance equations at these nodes.

The free body diagram of mass M1

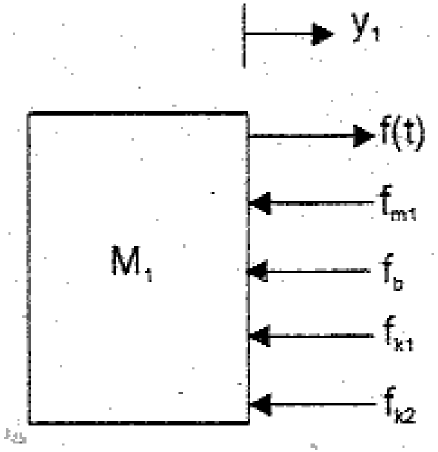


Fig 2.

The free body diagram of mass M2

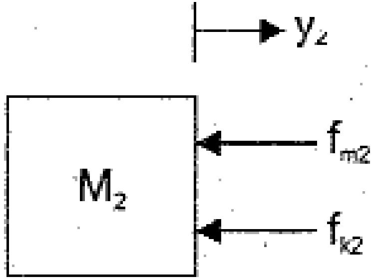


Fig 3.

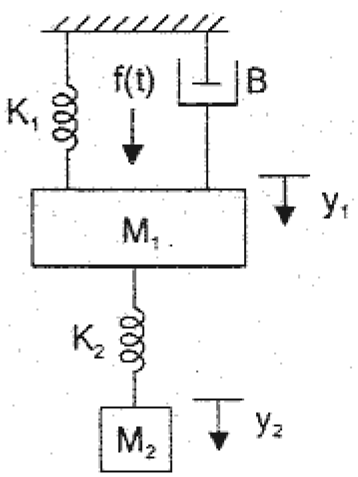


Fig 1.

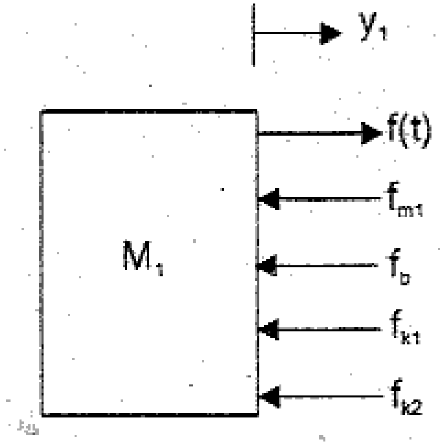


Fig 2.

The free body diagram of mass M_1 is shown in fig 2.

The opposing forces are marked as f_{m1}, f_b, f_{k1} and f_{k2}

$$f_{m1} = M_1 \frac{d^2y_1}{dt^2} \;\; ; \;\; f_b = B\frac{dy_1}{dt} \;\; ; \;\; f_{k1} = K_1 y_1 \;\; ; \;\; f_{k2} = K_2(y_1 - y_2)$$

By Newton's second law, $f_{m1} + f_b + f_{k1} + f_{k2} = f(t)$

$$\therefore M_1 \frac{d^2y_1}{dt^2} + B\frac{dy_1}{dt} + K_1 y_1 + K_2(y_1 - y_2) = f(t) \qquad .....(1)$$

On taking Laplace transform of equation (1) with zero initial conditions we get

$$\therefore M_1 \frac{d^2 y_1}{dt^2} + B \frac{dy_1}{dt} + K_1 y_1 + K_2(y_1 - y_2) = f(t) \quad .....(1)$$

$$M_1 s^2 Y_1(s) + BsY_1(s) + K_1 Y_1(s) + K_2[Y_1(s) - Y_2(s)] = F(s)$$

$$Y_1(s)[M_1 s^2 + Bs + (K_1 + K_2)] - Y_2(s)K_2 = F(s) \quad ...2$$

The free body diagram of mass M2 is show in Fig 2. The opposing forces acting on mass M2 are marked as f_{m2} and f_{k2}

$$f_{m2} = M_2 \frac{d^2 y_2}{dt^2} \;; \qquad f_{k2} = K_2(y_2 - y_1)$$

By Newton's second law, $f_{m2} + f_{k2} = 0$

$$\therefore M_2 \frac{d^2 y_2}{dt^2} + K_2(y_2 - y_1) = 0$$

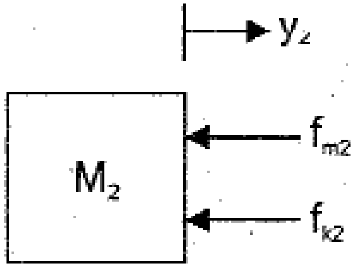


Fig 3.

$$\therefore M_2 \frac{d^2y_2}{dt^2} + K_2(y_2 - y_1) = 0$$

On taking Laplace transform of above equation with zero initial conditions we get

$$M_2 s^2 Y_2(s) + K_2[Y_2(s) - Y_1(s)] = 0$$

$$Y_2(s)\,[M_2 s^2 + K_2] - Y_1(s)\,K_2 = 0$$

$$\therefore Y_1(s) = Y_2(s)\,\frac{M_2 s^2 + K_2}{K_2} \qquad ...3$$

Substituting for Y1(s) from equation (3) in equation (2) we get

$$Y_2(s)\left[\frac{M_2 s^2 + K_2}{K_2}\right]\left[M_1 s^2 + Bs + (K_1 + K_2)\right] - Y_2(s)\,K_2 = F(s)$$

$$Y_2(s)\left[\frac{(M_2 s^2 + K_2)\,[M_1 s^2 + Bs + (K_1 + K_2)] - K_2^2}{K_2}\right] = F(s)$$

$$\therefore \frac{Y_2(s)}{F(s)} = \frac{K_2}{\left[M_1 s^2 + Bs + (K_1 + K_2)\right]\left[M_2 s^2 + K_2\right] - K_2^2}$$

RESULT

The differential equations governing the system are,

1. $$M_1\frac{d^2y_1}{dt^2}+B\frac{dy_1}{dt}+K_1y_1+K_2(y_1-y_2)=f(t)$$

2. $$M_2\frac{d^2y_2}{dt^2}+K_2(y_2-y_1)=0$$

The transfer function of the system is,

$$\frac{Y_2(s)}{F(s)}=\frac{K_2}{\left[M_1s^2+Bs+\ (K_1+K_2)\right]\left[M_2s^2+K_2\right]-K_2^2}$$

1. In mechanical translational system, the differential equations governing the system are obtained by writing force balance equations at nodes in the system. The nodes are meeting point of elements. Generally the nodes are 'Mass elements' in the system. In some cases the nodes may be without mass element.

2. The linear displacement of the masses (nodes) are assumed as X1, X2, X3, etc., and assign a displacement to each mass(node). The first derivative of the displacement is velocity and the second derivative of the displacement is acceleration

Equivalent Mechanical System (Node Basis)

While drawing analogous networks, it is always better to draw the equivalent mechanical system from the given Physical system. To draw such system use following steps:

- Step 1: Due applied force, identify the displacements in the mechanical system.
- Step 2: Identify the elements which are under the influence of different displacements.
- Step 3: Represent each displacement by a separate node, using Nodal Analysis.
- Step 4 : Show all the elements in parallel under the respective nodes which are under the influence of respective displacements.
- Step 5: Elements causing same change in displacement will get connected in parallel in between the respective nodes.

Remarks on Nodal Method

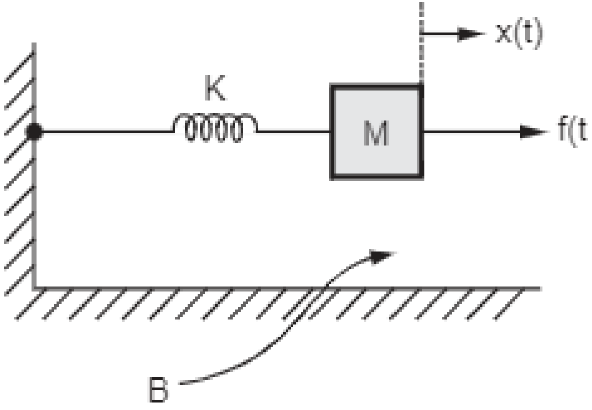


Fig (a) Physical system

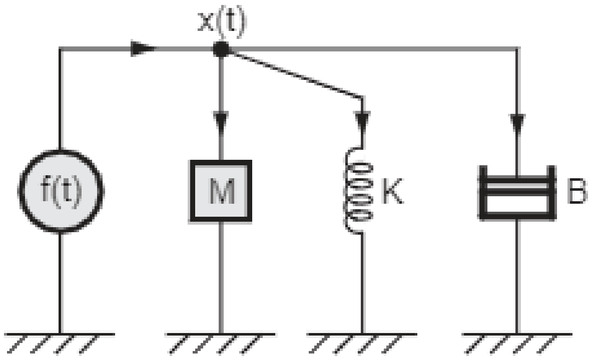


Fig (b) Equivalent mechanical system

Applied force f(t) is the input and displacement x is the output.

Mass 'M' will displaced by amount 'X' and as spring is connected to fixed support and friction 'B' is also with respect to fixed support, both K and B will be under influence of 'x' only. Now its equivalent system will contain one node and as all elements are under influence of x(t) alone, must be connected in parallel under that node.

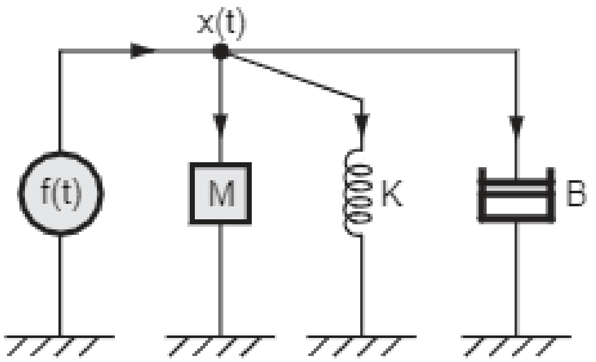


Fig (b) Equivalent mechanical system

write one differential equation by equating the sum of applied forces to the sum of opposing forces at Nodes or $\sum_{all} F = 0$

- The equilibrium equation will be,

$$f(t) = M \frac{d^2 x(t)}{dt^2} + K\,x(t) + B \frac{dx(t)}{dt}$$

Taking Laplace, $F(s) = Ms^2 X(s) + K X(s) + Bs X(s)$

$$= X(s)\left[Ms^2 + K + Bs\right]$$

Remarks on Nodal Method

a) The terms for an element connected to a node `x' and stationary surface (reference) is,

$$\text{For mass} \rightarrow M \frac{d^2x}{dt^2}$$

$$\text{For friction} \rightarrow B \frac{dx}{dt}$$

$$\text{For spring} \rightarrow Kx$$

b) The term for an element connected between the two nodes `$x_1$' and `x_2' i.e. between two moving surfaces is,

$$\text{For friction} \rightarrow B\left[\frac{dx_1}{dt} - \frac{dx_2}{dt}\right]$$

$$\text{For spring} \rightarrow K\left[x_1 - x_2\right]$$

Always the opposing force acts in a direction opposite to applied force

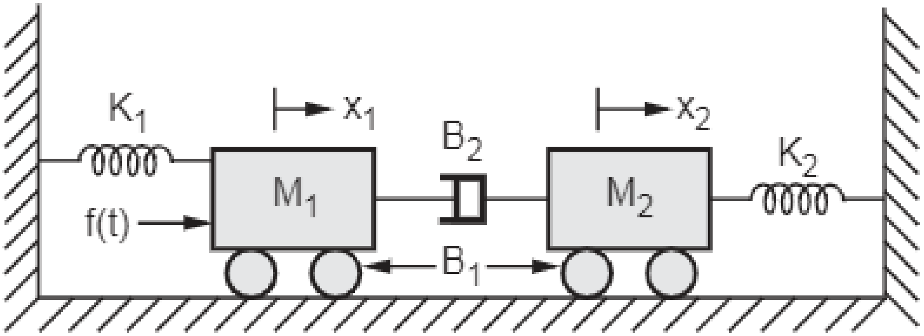
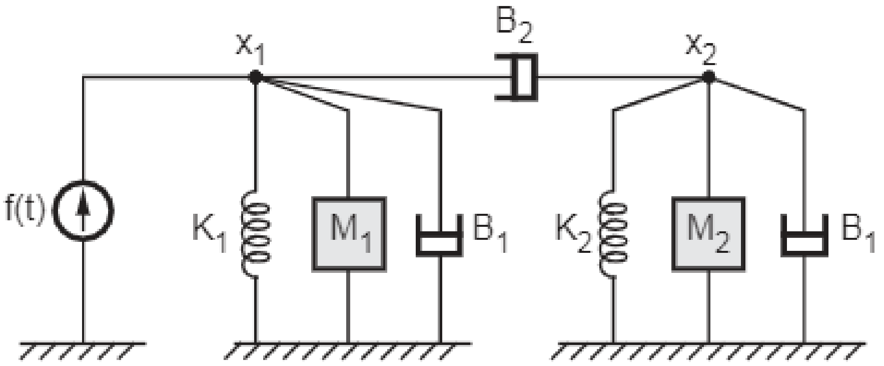


Figure. (a) Translational system,



Figure(b) Equivalent mechanical network

➢M1, K1 and B1 are under influence of displacement X1, as K1 and B1 are with respect to fixed support.

➢B2 is between X1 and X2.

➢M2, K2 and B1 are under X2, as K2 and B1 are with respect to fixed support.

The equivalent mechanical system is shown in the Fig(B).

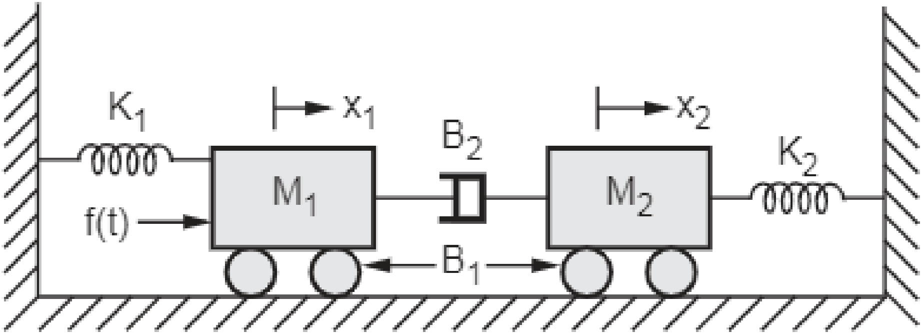
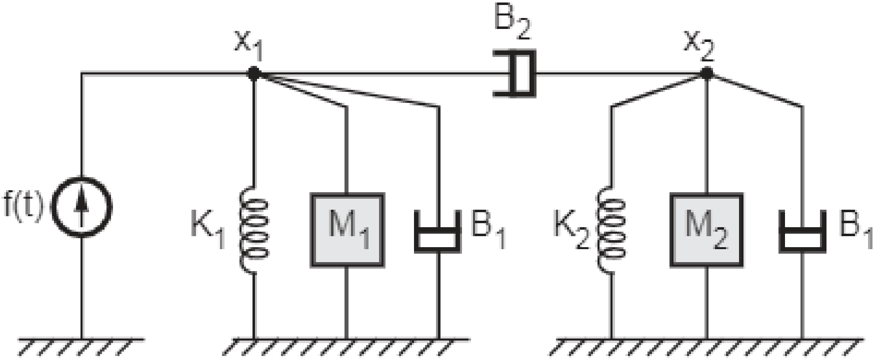


Figure. (a) Translational system,



Figure(b) Equivalent mechanical network

Remarks on Nodal Method

a) The terms for an element connected to a node `x' and stationary surface (reference) is,

For mass $\rightarrow M \dfrac{d^2x}{dt^2}$

For friction $\rightarrow B \dfrac{dx}{dt}$

For spring $\rightarrow Kx$

b) The term for an element connected between the two nodes `$x_1$' and `x_2' i.e. between two moving surfaces is,

For friction $\rightarrow B\left[\dfrac{dx_1}{dt} - \dfrac{dx_2}{dt}\right]$

For spring $\rightarrow K\,[\,x_1 - x_2]$

- The equilibrium equations at two nodes are,

$$f(t) = M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + K_1\, x_1 + B_2 \frac{d\,(x_1 - x_2)}{dt}$$

$$0 = B_2 \frac{d\,(x_2 - x_1)}{dt} + M_2 \frac{d^2x_2}{dt^2} + B_1 \frac{dx_2}{dt} + K_2\, x_2$$

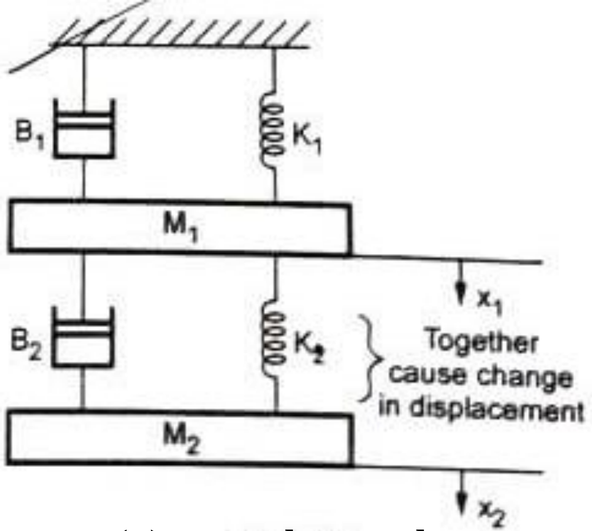
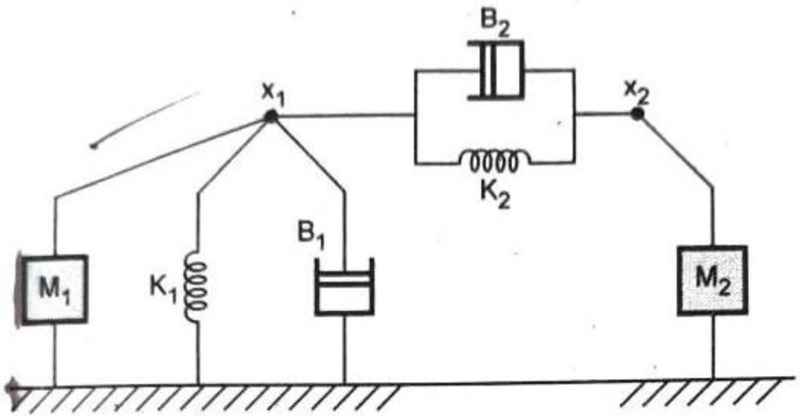


Figure. (a) Translational system,

(X1) – M1, B1, K1

(X1, X2) – B2, K2

(X2) – M2



Figure(b) Equivalent mechanical network

- In this below system M1 , B1 and K1 all are under the influence of displacement X1. This is because all are connected to rigid support.
- While there is change from x1 to x2 due to simultaneous effect of B2 and K2. So B2 and K2 are under the influence of (x1 - x2).
- But mass M2 is under the influence of X2 alone. Mass cannot be under the influence of difference between displacements.
- So in equivalent system the elements B1 , K1, and M1, all in parallel under x1, while B2 and K2 in parallel between X1 and X2 and element M2 is under node X2 as shown below

Consider the mechanical system shown in Figure (a).

It is simply a mass M attached to a spring (stiffness K) and a dash pot (viscous friction coefficient f) on which the force F acts. Displacement x is positive in the direction shown.

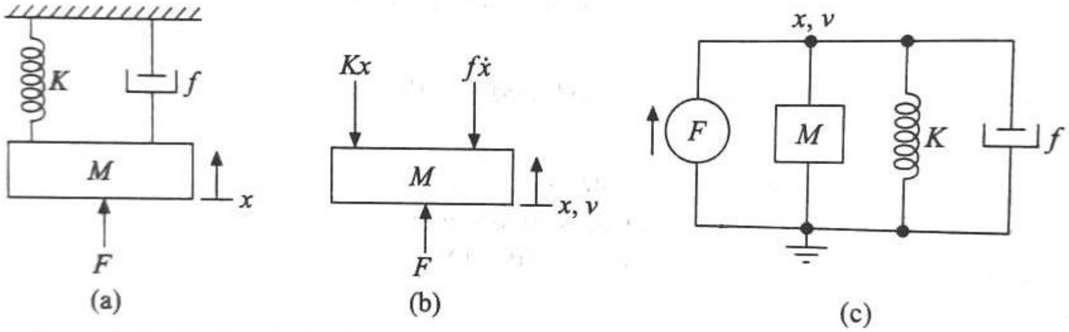


Figure. (a) Translational system, (b) free-body diagram, and (c) mechanical network.

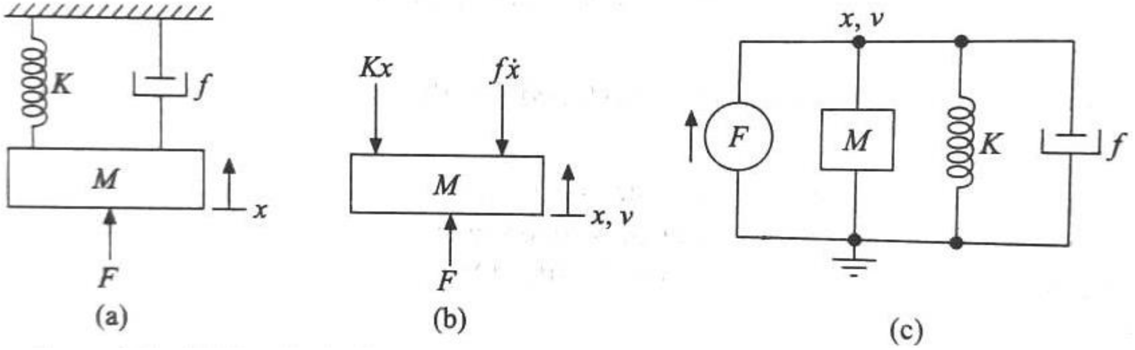
The systematic way of analyzing such a system is to draw a free-body diagram or mechanical network as shown in Figures (b) and (c) respectively. Then, by applying Newton's law of motion to the free-body diagram or mechanical network, the force equation can be written in terms of displacement x or velocity v as follows

$$F = F_m + F_b + F_k$$

$$\Rightarrow F = M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx$$

The free-body diagram is drawn like this: the free-body diagram for a translational system indicates the elements masses M and all the forces acting on them.

For the system shown in Figure (a), there is only one mass M. An external force F is acting on it to move it by a distance x or with a velocity v. This motion is opposed by the force components of mass M, f_m, and the force components due to spring K, f_k and damper, i.e. fk. So, the free-body diagram is as shown in Figure (b).



$$F = F_m + F_b + F_k$$

$$\Rightarrow F = M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx$$

The mechanical network for the given translational system of Figure (a) is drawn like this: one end of F is to be grounded. Also one end of M is to be grounded. So one end of F and M is grounded. The other end of F is connected to the other end of M because F is applied on M. In the mechanical system, one end of K and f is grounded and the other ends of K and f are connected to the free end of M. So in the mechanical network also one end of K and f is grounded and the other ends are connected to the free end of M.

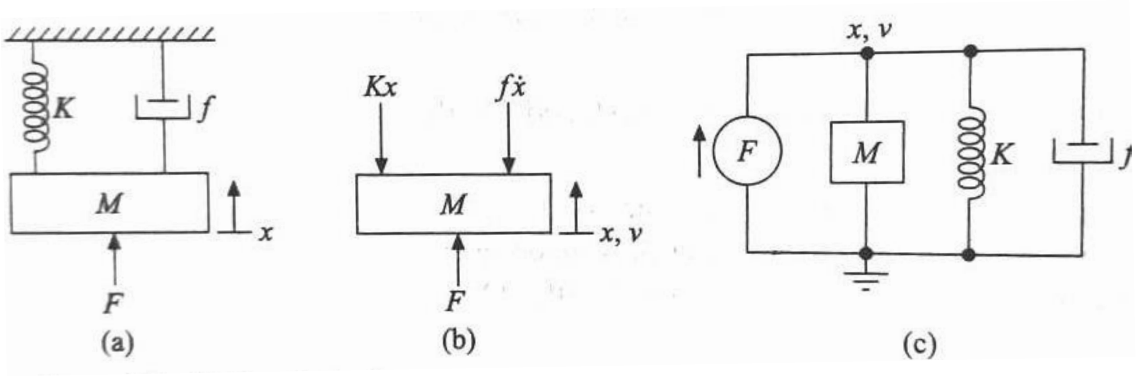


Figure. (a) Translational system, (b) free-body diagram, and (c) mechanical network.

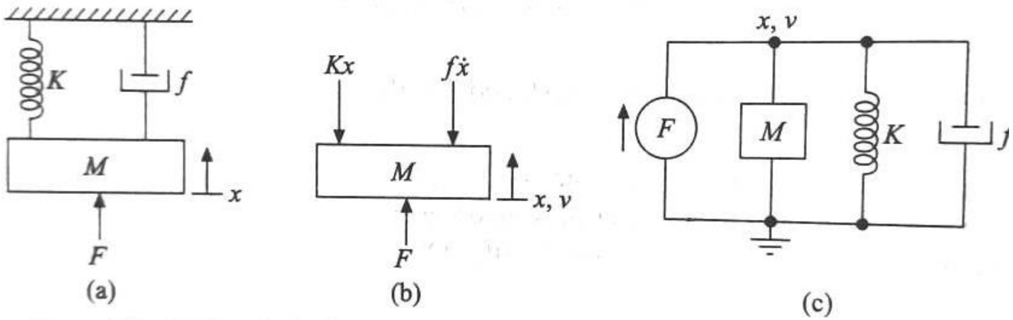


Figure. (a) Translational system, (b) free-body diagram, and (c) mechanical network.

Once a mechanical network is drawn, the differential equations can be written very easily at each node assuming that the displacement or velocity at the node under consideration is higher than the displacement or velocity at all other nodes. In Figure a. only one node is there, the other node is the reference or ground node. Only external force F is towards the node. All other forces, i.e. f_m, fb and fk are directed away from the node. Now the differential equation is like KCL equation for electrical circuits which says that the sum of currents coming into the node is equal to the sum of currents going out of the node. The force components are similar to currents. Therefore the force F(t) directed towards the node is equal to the sum of the force components f_m, fb and fk directed away from the node. Therefore, the equation $\Rightarrow F = M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx$

Modeling of Rotational Mechanical Systems

Rotational mechanical systems move about a fixed axis.

These systems mainly consist of three basic elements. Those are **moment of inertia, torsional spring** and **dashpot**.

If a torque is applied to a rotational mechanical system, then it is opposed by opposing torques due to moment of inertia, elasticity and friction of the system. Since the applied torque and the opposing torques are in opposite directions, the algebraic sum of torques acting on the system is zero.

MECHANICAL ROTATIONAL SYSTEM

LIST OF SYMBOLS USED IN MECHANICAL ROTATIONAL SYSTEM

θ = Angular displacement, rad

$\frac{d\theta}{dt}$ = Angular velocity, rad/sec

$\frac{d^2\theta}{dt^2}$ = Angular acceleration, rad/sec^2

T = Applied torque, N-m

J = Moment of inertia, Kg-m^2/rad

B = Rotational frictional coefficient, N-m/(rad/sec)

K = Stiffness of the spring, N-m/rad

Analogous elements

Sl. No.	Mechanical Translational Motion	Mechanical Rotational Motion
1	Mass (M)	Moment of inertia (J)
2	Damping coefficient /Friction (B)	Rotational Damping coefficient (B)
3	Spring constant or stiffness (K)	Torsional spring constant / stiffness (K)
4	Force (F)	Torque (T)
5	Displacement (x)	Angular displacement (θ)
6	Velocity (v=dx/dt)	Angular velocity (ω=dθ/dt)
7	Acceleration $(a = \frac{d^2x}{dt^2})$	Angular acceleration $(\alpha = \frac{d^2\theta}{dt^2})$

Torque Balance equation of idealized elements

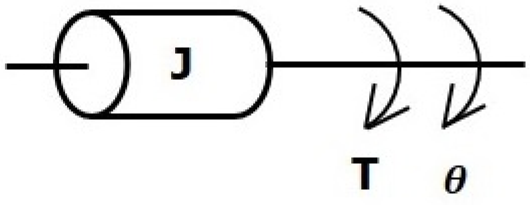
Let us now see the torque opposed by these three elements individually.

Torque Balance equation of idealized elements

Moment of Inertia

In translational mechanical system, mass stores kinetic energy. Similarly, in rotational mechanical system, moment of inertia stores **kinetic energy**.

If a torque is applied on a body having moment of inertia **J**, then it is opposed by an opposing torque due to the moment of inertia. This opposing torque is proportional to angular acceleration of the body. Assume elasticity and friction are negligible



Ideal rotational Mass element

$$T_j \propto \alpha$$

$$T = T_j = J\frac{d^2\theta}{dt^2}$$

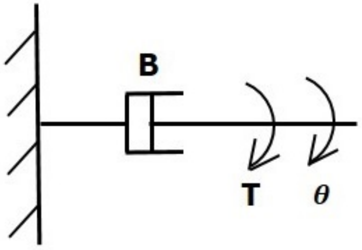
By Newton’s second law $T = T_j = J\frac{d^2\theta}{dt^2}$

Where,

- **T** is the applied torque
- T_j is the opposing torque due to moment of inertia
- **J** is moment of inertia
- **α** is angular acceleration
- **θ** is angular displacement

Rotational Dashpot

If a torque is applied on dashpot **B**, then it is opposed by an opposing torque due to the **rotational friction** of the dashpot. This opposing torque is proportional to the angular velocity of the body. Assume the moment of inertia and elasticity are negligible.



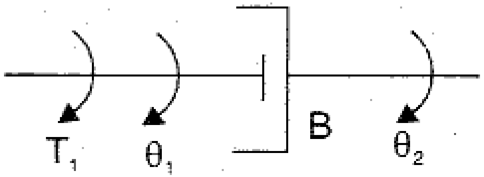
Ideal rotational dash-pot with one end fixed to reference

$$T_b \propto \omega$$

$$\Rightarrow T_b = B\omega = B\frac{d\theta}{dt}$$

$$T = T_b = B\frac{d\theta}{dt}$$

- T_b is the opposing torque due to the rotational friction of the dashpot
- B is the rotational friction coefficient
- ω is the angular velocity
- θ is the angular displacement



Ideal rotational dash-pot angular displacement at both end

Opposing torque is proportional to differential angular velocity

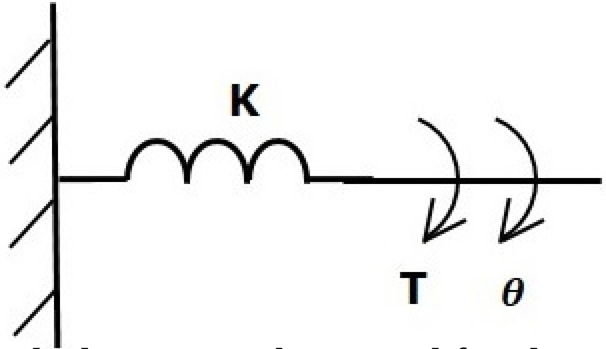
$$T_b \propto \frac{d}{dt}(\theta_1 - \theta_2) \quad \text{or} \quad T_b = B\frac{d}{dt}(\theta_1 - \theta_2)$$

$$\boxed{\therefore T = T_b = B\frac{d}{dt}(\theta_1 - \theta_2)}$$

Torsional Spring

In translational mechanical system, spring stores potential energy. Similarly, in rotational mechanical system, torsional spring stores **potential energy**.

If a torque is applied on torsional spring **K**, then it is opposed by an opposing torque due to the elasticity of torsional spring. This opposing torque is proportional to the angular displacement of the torsional spring. Assume that the moment of inertia and friction are negligible.



Ideal spring with one end fixed to reference

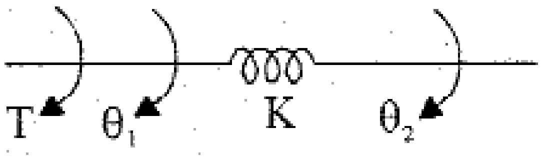
$$T_k \propto \theta$$

$$\Rightarrow T_k = K\theta$$

$$T = T_k = K\theta$$

- **T** is the applied torque
- T_k is the opposing torque due to elasticity of torsional spring
- **K** is the torsional spring constant
- **θ** is angular displacement

Torsional Spring



Ideal spring with angular displacement at both end

$$T_k \propto (\theta_1 - \theta_2) \quad \text{or} \quad T_k = K(\theta_1 - \theta_2)$$

$$\therefore \boxed{T = T_k = K(\theta_1 - \theta_2)}$$

Opposing torque is proportional to the differential angular displacement of the torsional spring

- **T** is the applied torque
- $\mathbf{T_k}$ is the opposing torque due to elasticity of torsional spring
- **K** is the torsional spring constant
- $\boldsymbol{\theta}$ is angular displacement

Guideline to determine the Transfer Function of Mechanical Rotational System

1. In mechanical rotational system, the differential equations governing the system are obtained by writing torque balance equations at nodes in the system. The nodes are meeting point of elements. Generally the nodes are mass elements with moment of inertia in the system. In some cases the nodes may be without mass element..

2. The angular displacement of the moment of inertia of the masses (nodes) are assumed as θ1,, θ2, θ3, etc., and assign a displacement to each mass (node). The first derivative of angular displacement is angular velocity and the second derivative of the angular displacement is angular acceleration.

Guideline to determine the Transfer Function of Mechanical Rotational System

3. Draw the free body diagrams of the system. The free body diagram is obtained by drawing each moment of inertia of mass separately and then marking all the torques acting on that body. Always the opposing torques acts in a direction opposite to applied torque

4. The mass has to rotate in the direction of the applied torque. Hence the angular displacement, velocity and acceleration of the mass will be in the direction of the applied torque. If there is no applied torque then the angular displacement, velocity and acceleration of the mass is in a direction opposite to that of opposing torque.

Guideline to determine the Transfer Function of Mechanical Rotational System

5. For each free body diagram write one differential equation by equating the sum of applied torques to the sum of opposing torques.

6. Take Laplace transform of differential equation to convert them to algebraic equations. Then rearrange the s-domain equations to eliminate the unwanted variables and obtain the relation between output variable and input variable. This ratio is the transfer function of the system,

Guideline to determine the Transfer Function of Mechanical Rotational System

Note :

Laplace transform of $\theta = \mathcal{L}\{\theta\} = \theta(s)$

Laplace transform of $\dfrac{d\theta}{dt} = \mathcal{L}\left\{\dfrac{d\theta}{dt}\right\} = s\,\theta(s)$ (with zero initial conditions)

Laplace transform of $\dfrac{d^2\theta}{dt^2} = \mathcal{L}\left\{\dfrac{d^2\theta}{dt^2}\right\} = s^2\,\theta(s)$ (with zero initial conditions)

Write the differential equations governing the mechanical rotational system shown in fig 1. transfer function of the system.

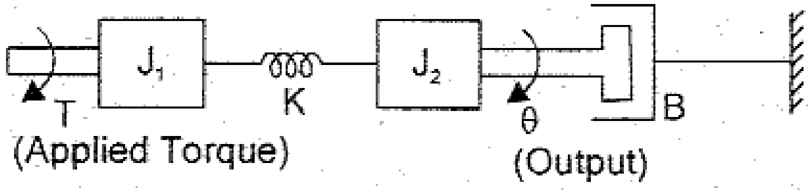


Fig 1.

In the given system, applied torque T is the input and angular displacement θ is the output.

Let, Laplace transform of $T = \mathcal{L}\{T\} = T(s)$

Laplace transform of $\theta = \mathcal{L}\{\theta\} = \theta(s)$

Laplace transform of $\theta_1 = \mathcal{L}\{\theta_1\} = \theta_1(s)$

Hence the required transfer function is $\dfrac{\theta(s)}{T(s)}$

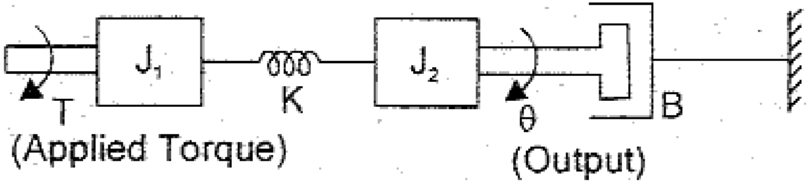


Fig 1.

The system has two nodes and they are masses with moment of inertia J1 and J2. The differential equations governing the system are given by torque balance equations at these nodes.

Let the angular displacement of mass with moment of inertia J1 be θ1. The free body diagram of J1 is shown in fig 2. The opposing torques acting on J1 are marked as Tj1 and Tk.

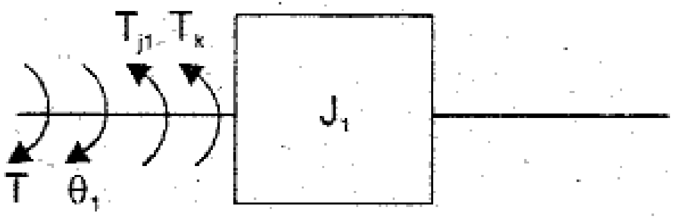


Fig 2 : *Free body diagram of mass with moment of inertia J_1.*

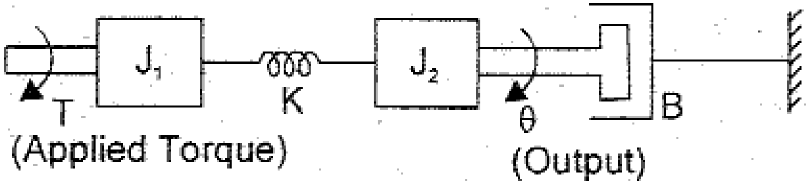


Fig 1.

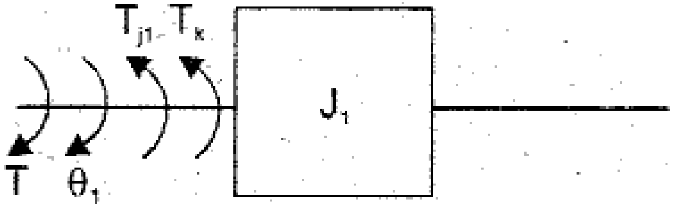


Fig 2 : Free body diagram of mass with moment of inertia J_1.

$$T_{j1} = J_1 \frac{d^2\theta_1}{dt^2} \quad ; \quad T_k = K(\theta_1 - \theta)$$

By Newton's second law, $T_{j1} + T_k = T$

$$J_1 \frac{d^2\theta_1}{dt^2} + K(\theta_1 - \theta) = T$$

$$J_1 \frac{d^2\theta_1}{dt^2} + K\theta_1 - K\theta = T \qquad \ldots(1)$$

On taking Laplace transform of above equation(1) with zero initial condition we get,

$$J_1 s^2 \theta_1(s) + K\theta_1(s) - K\theta(s) = T(s)$$

$$(J_1 s^2 + K)\,\theta_1(s) - K\,\theta(s) = T(s) \qquad \ldots(2)$$

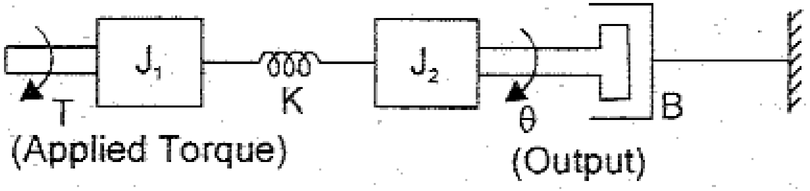


Fig 1.

The freebody diagram of mass with moment of inertia J2 is shown in fig 3. The opposing torques acting on J2,are marked as Tj2, Tb and Tk

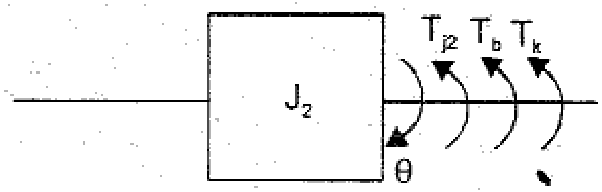


Fig 3 : Free body diagram of mass with moment of inertia J_2.

$$T_{j2} = J_2 \frac{d^2\theta}{dt^2} \quad ; \quad T_b = B\frac{d\theta}{dt} \quad ; \quad T_k = K(\theta - \theta_1)$$

By Newton's second law, $T_{j2} + T_b + T_k = 0$

$$\therefore J_2 \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K(\theta - \theta_1) = 0$$

$$J_2 \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta - K\theta_1 = 0$$

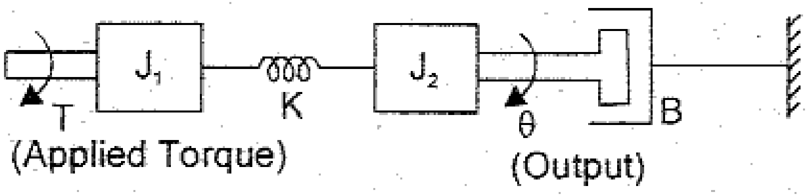


Fig 1.

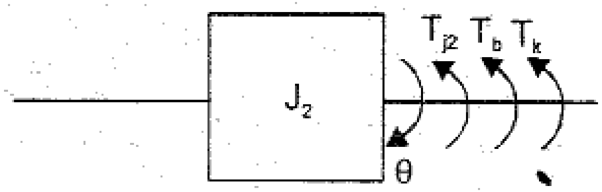


Fig 3 : Free body diagram of mass with moment of inertia J_2.

$$J_2 \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta - K\theta_1 = 0$$

On taking Laplace transform of above equation with zero initial condition we get,

$$J_2 s^2\theta(s) + B s\,\theta(s) + K\theta(s) - K\theta_1(s) = 0$$

$$(J_2 s^2 + Bs + K)\,\theta(s) - K\theta_1(s) = 0$$

$$\theta_1(s) = \frac{(J_2 s^2 + Bs + K)}{K}\,\theta(s) \qquad \ldots(3)$$

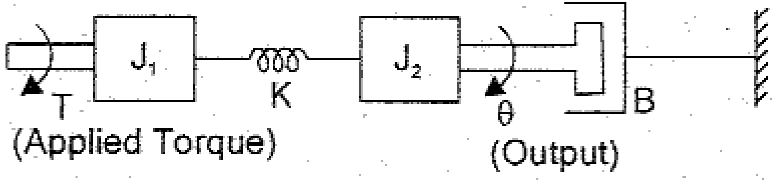


Fig 1.

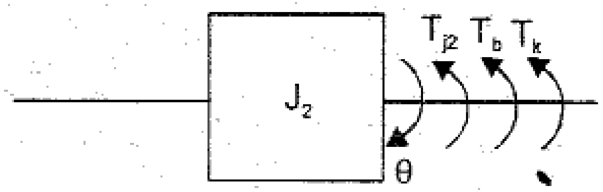


Fig 3 : *Free body diagram of mass with moment of inertia J_2.*

Substituting for θ1(s) from equation (3) in equation (2) we get,

$$(J_1 s^2 + K)\,\frac{(J_2 s^2 + Bs + K)}{K}\,\theta(s) - K\theta(s) = T(s)$$

$$\left[\frac{(J_1 s^2 + K)\,(J_2 s^2 + Bs + K) - K^2}{K}\right]\theta(s) = T(s)$$

$$\therefore \frac{\theta(s)}{T(s)} = \frac{K}{(J_1 s^2 + K)\,(J_2 s^2 + Bs + K) - K^2}$$

RESULT

The differential equations governing the system are

1. $J_1 \frac{d^2\theta_1}{dt^2} + K\theta_1 - K\theta = T$

2. $J_2 \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta - K\theta_1 = 0$

The transfer function of the system is

$$\frac{\theta(s)}{T(s)} = \frac{K}{(J_1 s^2 + K)(J_2 s^2 + Bs + K) - K^2}$$

Write the differential equations governing the mechanical rotational system shown in fig 1. transfer function of the system.

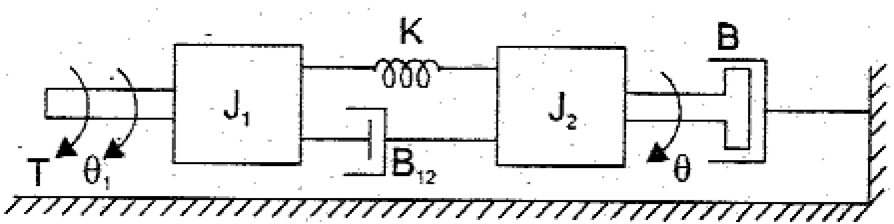


Fig 1.

In the given system, applied torque T is the input and angular displacement θ is the output.

Let, Laplace transform of $T = \mathcal{L}\{T\} = T(s)$

Laplace transform of $\theta = \mathcal{L}\{\theta\} = \theta(s)$

Laplace transform of $\theta_1 = \mathcal{L}\{\theta_1\} = \theta_1(s)$

Hence the required transfer function is $\dfrac{\theta(s)}{T(s)}$

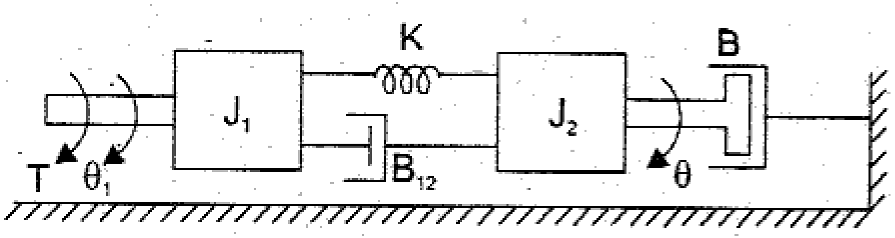


Fig 1.

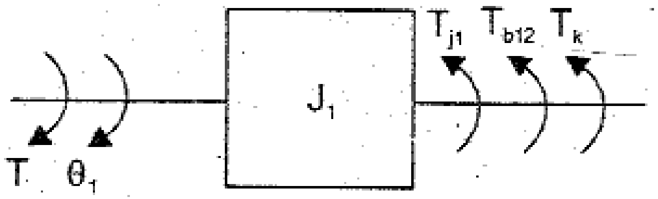


Fig 2 : Free body diagram of mass with moment of inertia J_1.

The system has two nodes and they are masses with moment of inertia J1 and J2. The differential equations governing the system are given by torque balance equations at these nodes.

Let the angular displacement of mass with moment of inertia J1 be θ1. The free body diagram of J1 is shown in fig 2. The opposing torques acting on J1 are marked as Tj1, Tb12 and Tk.

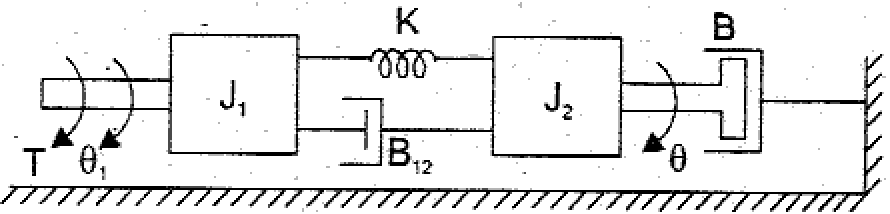


Fig 1.

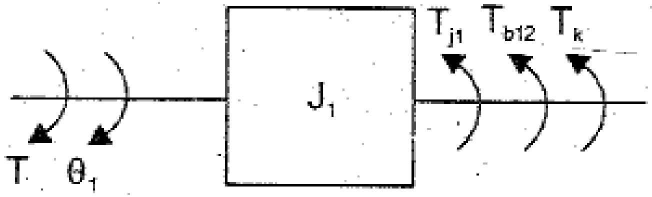


Fig 2 : *Free body diagram of mass with moment of inertia J_1.*

$$T_{j1} = J_1 \frac{d^2\theta_1}{dt^2} \quad ; \quad T_{b12} = B_{12}\frac{d}{dt}(\theta_1 - \theta) \quad ; \quad T_k = K(\theta_1 - \theta)$$

By Newton's second law, $T_{j1} + T_{b12} + T_k = T$

$$J_1 \frac{d^2\theta_1}{dt^2} + B_{12}\frac{d}{dt}(\theta_1 - \theta) + K(\theta_1 - \theta) = T$$

On taking Laplace transform of above equation with zero initial condition we get,

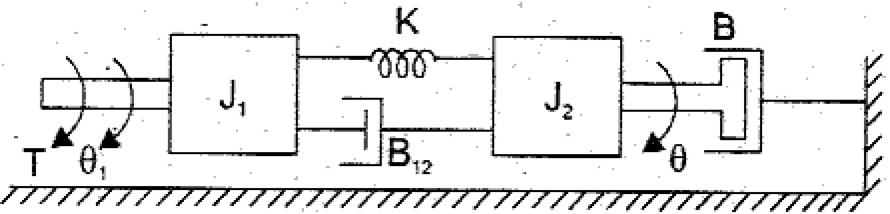


Fig 1.

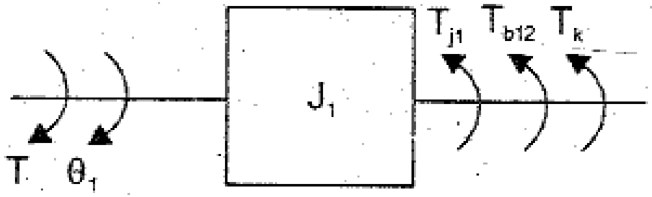


Fig 2 : Free body diagram of mass with moment of inertia J_1.

$$J_1 \frac{d^2\theta_1}{dt^2} + B_{12}\frac{d}{dt}(\theta_1 - \theta) + K(\theta_1 - \theta) = T$$

On taking Laplace transform of above equation with zero initial condition we get,

$$J_1 s^2 \theta_1(s) + s\,B_{12}\,[\theta_1(s) - \theta(s)] + K\theta_1(s) - K\theta(s) = T(s)$$

$$\theta_1(s)\,[J_1 s^2 + sB_{12} + K] - \theta(s)\,[sB_{12} + K] = T(s) \qquad \ldots(1)$$

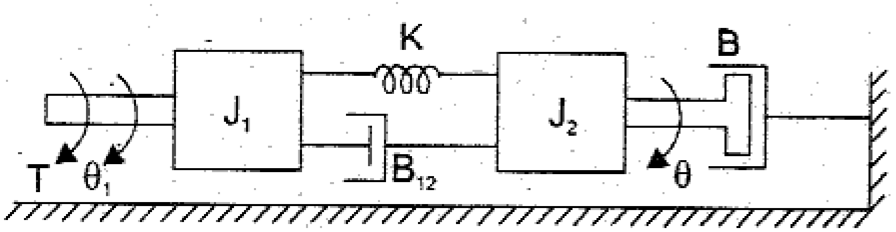


Fig 1.

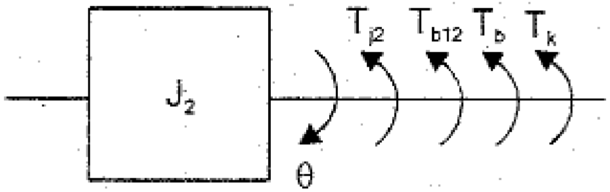


Fig 3 : Free body diagram of mass with moment of inertia J2

The freebody diagram of mass with moment of inertia J2 is shown in fig 3. The opposing torques are marked as Tj2, Tb12, Tb and Tk.

$$T_{j2} = J_2 \frac{d^2\theta}{dt^2} \quad ; \quad T_{b12} = B_{12} \frac{d}{dt}(\theta - \theta_1)$$

$$T_b = B \frac{d\theta}{dt} \quad ; \quad T_k = K(\theta - \theta_1)$$

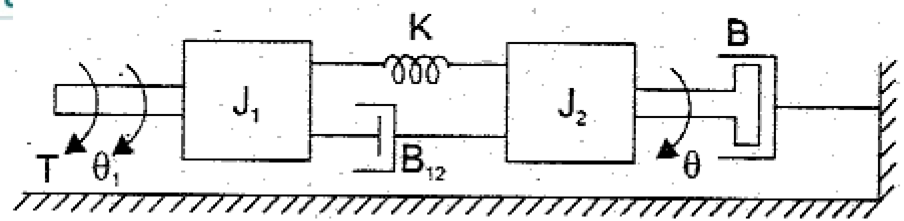


Fig 1.

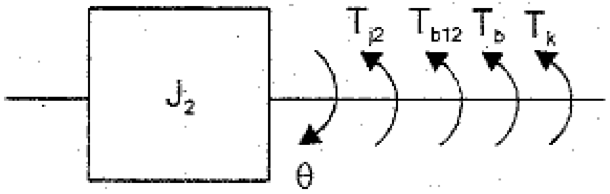


Fig 3 : Free body diagram of mass with moment of inertia J_2

$$J_2 \frac{d^2\theta}{dt^2} - B_{12}\frac{d\theta_1}{dt} + \frac{d\theta}{dt}(B_{12} + B) + K\theta - K\theta_1 = 0$$

On taking Laplace transform of above equation with zero initial condition we get,

$$J_2 s^2\theta(s) - B_{12}s\theta_1(s) + s\theta(s)\,[B_{12} + B] + K\theta(s) - K\theta_1(s) = 0$$

$$\theta(s)\,[s^2 J_2 + s(B_{12} + B) + K] - \theta_1(s)\,[sB_{12} + K] = 0$$

$$\theta_1(s) = \frac{[s^2 J_2 + s(B_{12} + B) + K]}{[sB_{12} + K]}\,\theta(s) \qquad \ldots(2)$$

Substituting for θ1(s) from equation (2) in equation (1) we get,

$$[J_1 s^2 + sB_{12} + K] \frac{[J_2 s^2 + s(B_{12} + B) + K]\,\theta(s)}{(sB_{12} + K)} - (sB_{12} + K)\,\theta(s) = T(s)$$

$$\left[\frac{(J_1 s^2 + sB_{12} + K)\,[J_2 s^2 + s(B_{12} + B) + K] - (sB_{12} + K)^2}{(sB_{12} + K)}\right]\theta(s) = T(s)$$

$$\therefore \frac{\theta(s)}{T(s)} = \frac{(sB_{12} + K)}{(J_1 s^2 + sB_{12} + K)\,[J_2 s^2 + s(B_{12} + B) + K] - (sB_{12} + K)^2}$$

RESULT

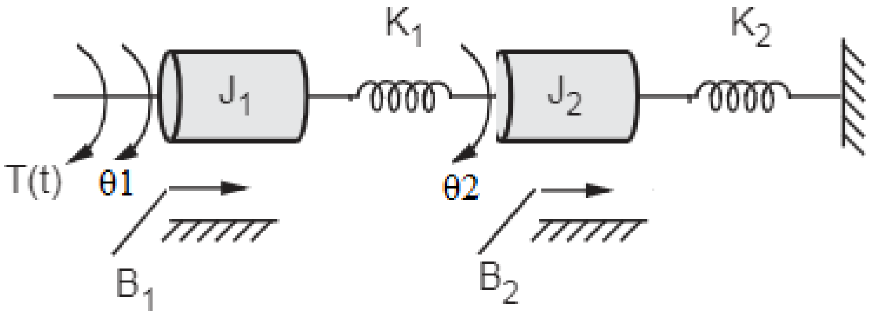
The differential equations governing the system are

1. $J_1 \frac{d^2\theta_1}{dt^2} + B_{12}\frac{d}{dt}(\theta_1 - \theta) + K(\theta_1 - \theta) = T$

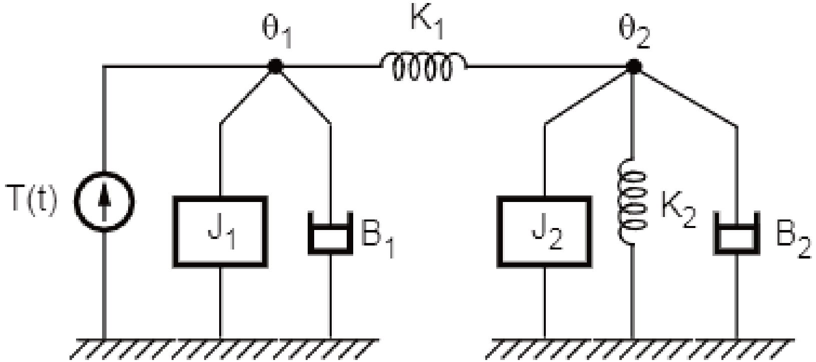
2. $J_2 \frac{d^2\theta}{dt^2} - B_{12}\frac{d\theta_1}{dt} + \frac{d\theta}{dt}(B_{12} + B) + K(\theta - \theta_1) = 0$

The transfer function of the system is

$$\frac{\theta(s)}{T(s)} = \frac{(sB_{12} + K)}{(J_1 s^2 + sB_{12} + K)\,[J_2 s^2 + s(B_{12} + B) + K] - (sB_{12} + K)^2}$$



- J1 and B1 are under Θ1.
- Spring K1 is between Θ1 and Θ2.
- J2, B2 and K2 are under Θ2.



The equivalent mechanical system

$$T(t) = J_1 \frac{d^2\theta_1}{dt^2} + B_1 \frac{d\theta_1}{dt} + K_1(\theta_1 - \theta_2) \quad \ldots (1)$$

$$0 = K_1(\theta_2 - \theta_1) + J_2 \frac{d^2\theta_2}{dt^2} + B_2 \frac{d\theta_2}{dt} + K_2\theta_2 \quad \ldots (2)$$

Electrical Analogies of Mechanical Systems

In between electrical and mechanical systems there exists a fixed analogy and their exists a similarity between their equilibrium equations. Due to this, it is possible to draw an electrical system which will behave exactly similar to the given mechanical system, this is called electrical analogous of given mechanical system and vice versa.

It is always advantageous to obtain electrical analogous of the given mechanical system as we are well familiar with the methods of analysing electrical network than mechanical systems: .

There are Two methods of obtaining electrical analogous networks, namely

1) Force - Voltage Analogy i.e. Direct Analogy.
2) Force - Current Analogy i.e. Inverse Analogy.

Electrical Analogous of Mechanical systems

Mechanical System		Electrical System	
Translational Mechanical System	**Rotational Mechanical System**	**Force/Torque-Voltage Analogy**	**Force/Torque-Current Analogy**
Force(F)	Torque(T)	Voltage(V)	Current(i)
Mass(M)	Moment of inertia(J)	Inductance(L)	Capacitance(C)
Frictional coefficient(B)	Rotational friction coefficient(B)	Resistance(R)	Reciprocal of Resistance(1/R)
Spring constant(K)	Torsional spring constant(K)	Reciprocal of Capacitance (1/c)	Reciprocal of Inductance(1/L)
Displacement(x)	Angular displacement(θ)	Charge(q)	Magnetic flux(Ø)
Velocity(v)	Angular velocity(ω)	Current(i)	Voltage(V)

Systems are said to be analogous if they satisfy the following two conditions.

- The two systems are physically different
- The differential equations governing the systems or transfer functions are in identical form

The three basic elements mass, dash-pot and spring that are used in modelling mechanical translational systems are analogous to resistance, inductance and capacitance of electrical systems

The input force in mechanical system is analogous to either voltage source or current source in electrical systems. The output velocity (first derivative of displacement) in mechanical system is analogous to either current or voltage in an element in electrical system.

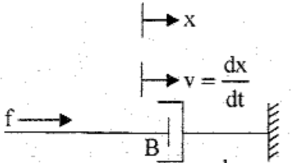
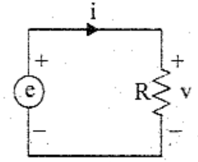
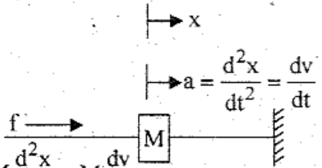
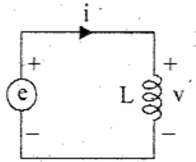
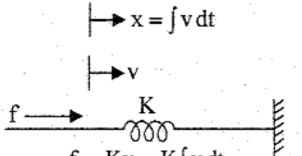
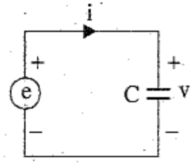
Since the electrical systems has two types of inputs either voltage or current source, there are two types of analogies :

1. Force-voltage analogy
2. Force-current analogy.

ANALOGOUS ELEMENTS OF FORCE-VOLTAGE ANALOGY

The force balance equations of mechanical elements and their analogous electrical elements in force-voltage analogy

Translational	Rotational	Electrical
Force	Torque T	Voltage V
Mass M	Inertia J	Inductance L
Friction constant B	Tortional friction constant B	Resistance R
Spring constant K N/m	Tortional spring constant K Nm/rad	Reciprocal of capacitor 1/C
Displacement 'x'	θ	Charge q
Velocity $\dot{x} = \frac{dx}{dt}$	$\dot{\theta} = \frac{d\theta}{dt} = \omega$	Current $i = \frac{dq}{dt}$

Mechanical system	Electrical system
Input : Force Output : Velocity	Input : Voltage source Output : Current through the element
 $f = B\frac{dx}{dt} = Bv$	 $e = v$ and $v = Ri$ $\therefore e = Ri$
 $f = M\frac{d^2x}{dt^2} = M\frac{dv}{dt}$	 $e = v$ and $v = L\frac{di}{dt}$ $\therefore e = L\frac{di}{dt}$
 $f = Kx = K\int v\,dt$	 $e = v$ and $v = \frac{1}{C}\int i\,dt$ $\therefore e = \frac{1}{C}\int i\,dt$

ANALOGOUS QUANTITIES OF FORCE-VOLTAGE ANALOGY

Item	Mechanical system	Electrical system (mesh basis system)
Independent variable (input)	Force, f	Voltage, e, v
Dependent variable (output)	Velocity, v	Current, i
	Displacement, x	Charge, q
Dissipative element	Frictional coefficient of dashpot, B	Resistance, R
Storage element	Mass, M	Inductance, L
	Stiffness of spring, K	Inverse of capacitance, 1/C
Physical law	Newton's second law $\sum f = 0$	Kirchoff 's voltage law $\sum v = 0$
Changing the level of independent variable	Lever $\frac{f_1}{f_2} = \frac{l_1}{l_2}$	Transformer $\frac{e_1}{e_2} = \frac{N_1}{N_2}$

Replacements for force-voltage analogy

$$F \rightarrow V, \quad M \rightarrow L, B \rightarrow R, \quad K \rightarrow 1/C, x \rightarrow q,$$

$$\dot{x} \rightarrow i \text{ (current)}, \; x \rightarrow \int i \, dt$$

The following points serve as guidelines to obtain electrical analogous of mechanical systems on force-voltage analogy.

1. In electrical systems the elements in series will have same current, likewise in mechanical systems, the elements having same velocity are said to be in series.

2. The elements having same velocity in mechanical system should have analogous same current in electrical analogous system.

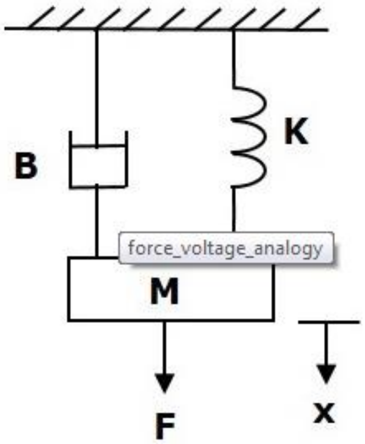
3. Each node (meeting point of elements) in the mechanical system corresponds to a closed loop in electrical system. A mass is considered as a node.

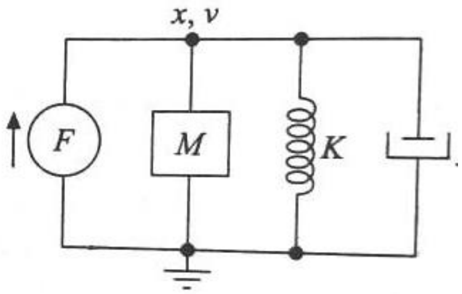
4. The number of meshes in electrical analogous is same as that of the number of nodes (masses) in mechanical system. Hence the number of mesh currents and system equations will be same as that of the number of nodes (masses) in mechanical system.

5. The mechanical driving sources (force) and passive elements connected to the node (mass) in mechanical system should be represented by analogous elements in a closed loop in analogous electrical system.

6. The element connected between two (nodes) masses in mechanical system is represented as a common element between two meshes in electrical analogous system.

ANALOGOUS QUANTITIES OF FORCE-VOLTAGE ANALOGY





$$F = F_m + F_b + F_k$$

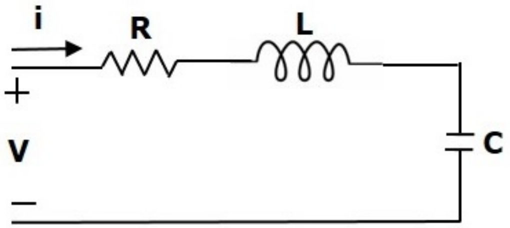
$$\Rightarrow F = M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx \qquad \ldots.(1)$$

Replacements for force-voltage analogy

$F \rightarrow V$, $M \rightarrow L$, $B \rightarrow R$, $K \rightarrow 1/C$, $x \rightarrow q$,

$\dot{x} \rightarrow i$ (current), $x \rightarrow \int i\, dt$

This circuit contains a resistor, an inductor and a capacitor. All these electrical elements are connected in a series. The input voltage applied to this circuit is V volts and the current flowing through the circuit is i Amps.



Mesh equation for this circuit is $V = Ri + L\frac{di}{dt} + \frac{1}{c}\int idt \qquad \ldots.(2)$

Substitute, $i = \frac{dq}{dt}$ in Equation 2

$$V = R\frac{dq}{dt} + L\frac{d^2q}{dt^2} + \frac{q}{C}$$

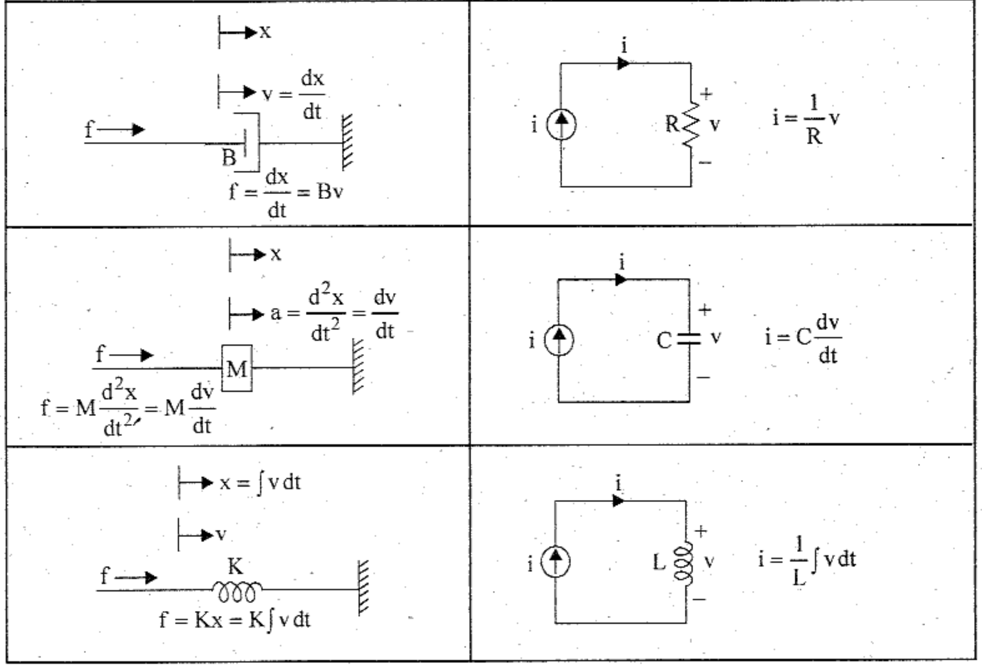
$$\Rightarrow V = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \left(\frac{1}{c}\right)q \qquad \ldots.(3)$$

ANALOGOUS ELEMENTS OF FORCE-CURRENT ANALOGY

Translational	Rotational	Electrical
F Force	T	Current I
M Mass	J	C
B friction	B	1/R
K Spring	K	1/L
x displacement	θ	ϕ
$\dot{x}$ Velocity $= \frac{dx}{dt}$	$\dot{\theta} = \frac{d\theta}{dt} = \omega$	Voltage 'v' $= \frac{d\phi}{dt}$

$F \rightarrow I$, $M \rightarrow C$, $B \rightarrow 1/R$, $K \rightarrow 1/L$, $x \rightarrow \phi$,
$\dot{x} = e(e.m.f.)$, $x \rightarrow \int e\, dt$

Mechanical system	Electrical system
Input : Force Output : Velocity	Input : Current source Output : Voltage across the element



ANALOGOUS QUANTITIES OF FORCE-CURRENT ANALOGY

Translational	Rotational	Electrical
F Force	T	Current I
M Mass	J	C
B friction	B	1/R
K Spring	K	1/L
x displacement	θ	ϕ
$\dot{x}$ Velocity $= \frac{dx}{dt}$	$\dot{\theta} = \frac{d\theta}{dt} = \omega$	Voltage 'v' $= \frac{d\phi}{dt}$

Item	Mechanical system	Electrical system (node basis system)
Independent variable (input)	Force, f	Current, i
Dependent variable (output)	Velocity, v	Voltage, v
	Displacement, x	Flux, ϕ
Dissipative element	Frictional coefficient of dashpot, B	Conductance G=1/R
Storage element	Mass, M	Capacitance, C
	Stiffness of spring, K	Inverse of inductance, 1/L
Physical law	Newton's second law $\sum f = 0$	Kirchoff 's current law $\sum i = 0$
Changing the level of independent variable	Lever $\frac{f_1}{f_2} = \frac{l_1}{l_2}$	Transformer $\frac{i_1}{i_2} = \frac{N_2}{N_1}$

ANALOGOUS QUANTITIES OF FORCE-CURRENT ANALOGY

Replacements for force-Current analogy

$F \rightarrow I, \quad M \rightarrow C, \quad B \rightarrow 1/R, \quad K \rightarrow 1/L, \quad x \rightarrow \phi,$

$\dot{x} = e(\text{e.m.f.}), \; x \rightarrow \int e \, dt$

Key Point The elements which are in series in F - V analogy, get connected in parallel in F - I analogous network and which are in parallel in F - V analogy, get connected in series in F - I analogous network.

The following points serve as guidelines to obtain electrical analogous of mechanical systems based on force-current analogy.

1. In electrical systems element in parallel will have same voltage, likewise in mechanical systems, the elements having same force are said to be in parallel.

2. The elements having same velocity in mechanical system should have analogous same voltage in electrical analogous system.

3. Each node (meeting point of elements) in the mechanical system corresponds to a node in electrical system. A mass is considered as a node

4. The number of nodes in electrical analogous is same as that of the number of nodes (masses)in mechanical system. Hence the number of node voltages and system equations will be same as that of the number of velocities of (nodes) masses in mechanical system.

5. The mechanical driving sources (forces) and passive elements connected to the node (mass)in mechanical system should be represented by analogous elements connected to a node in electrical system.

6. The element connected between two nodes (masses) in mechanical system is represented. as a common element between two nodes in electrical analogous system.

Write the differential equations governing the mechanical system shown in fig. Draw the force voltage and force-current electrical analogous circuits and verify by writing mesh and node equations.

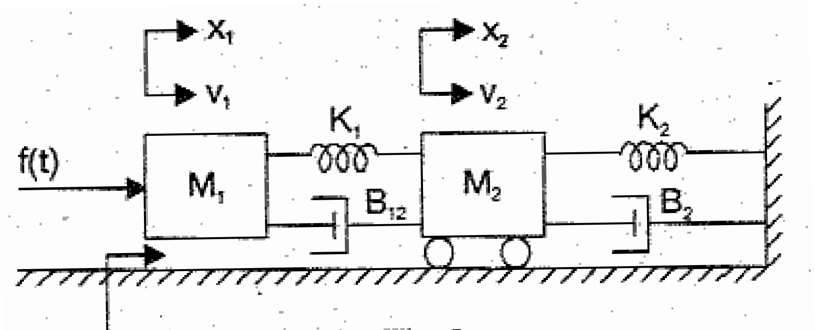


Fig 1.

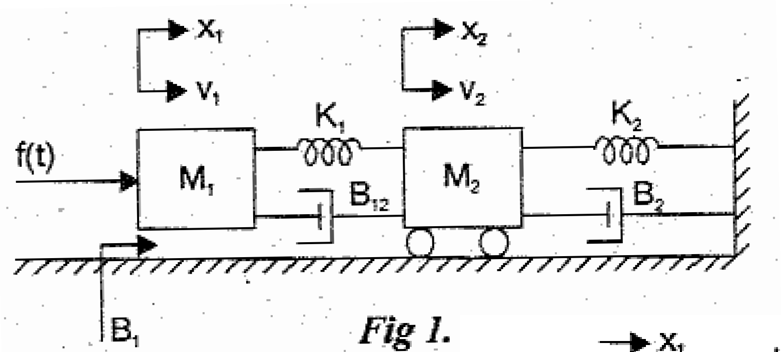
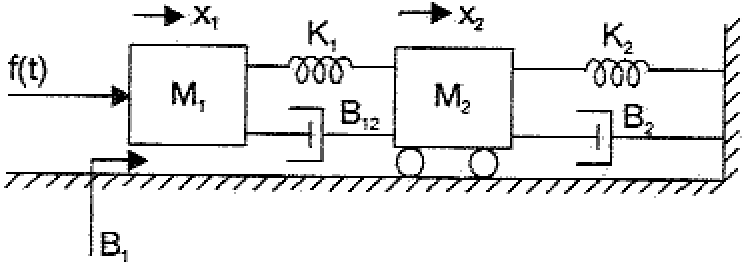


Fig 1.



The free body diagram of M1

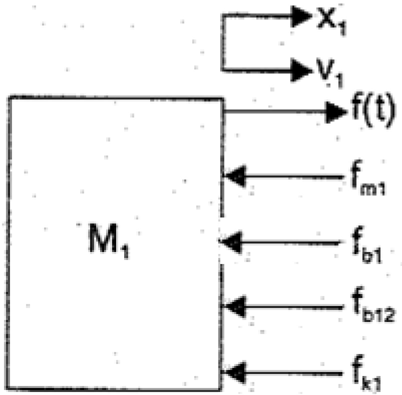


Fig 2.

By Newton's second law, $f_{m1} + f_{b1} + f_{b12} + f_{k1} = f(t)$

$$\therefore M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B_{12} \frac{d}{dt}(x_1 - x_2) + K_1(x_1 - x_2) = f(t) \qquad \text{......(1)}$$

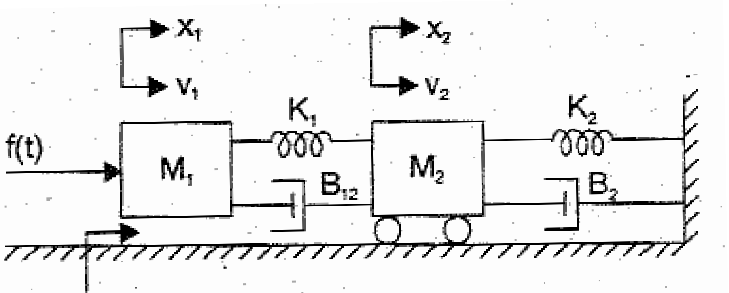


Fig 1.

The free body diagram of M2

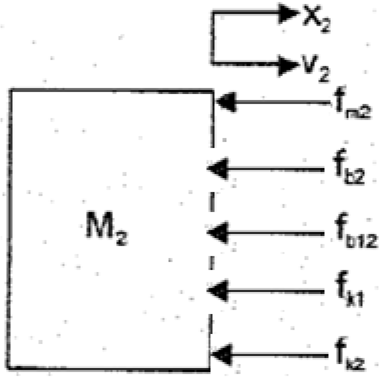


Fig 3.

By Newton's second law, $f_{m2} + f_{b2} + f_{k2} + f_{b12} + f_{k1} = 0$

$$M_2 \frac{d^2x_2}{dt^2} + B_2 \frac{dx_2}{dt} + K_2x_2 + B_{12} \frac{d}{dt}(x_2 - x_1) + K_1(x_2 - x_1) = 0 \qquad .....(2)$$

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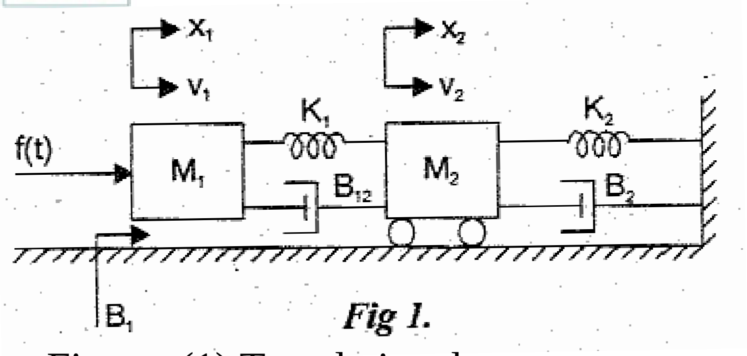
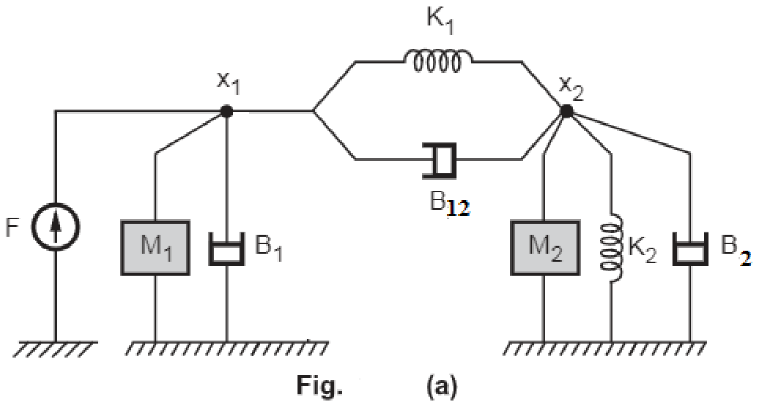


Figure. (1) Translational system

(X1) – M1, B1

(X1, X2) – K1, B12,

(X2) – M2, K2, B2



Figure(a) Equivalent mechanical network

- In this Above system M1 , B1 are under the influence of displacement X1. This is because all are connected to rigid support.
- B12 and K1 are under the influence of difference between displacements(x1 - x2).
- But mass M2, K2 & B2 is under the influence of X2 alone.
- Mass cannot be under the influence of difference between displacements.

- So in equivalent system the elements B1 and M1 in parallel under node x1, while B12 and K1 in parallel between node X1 and node X2 and element M2, K2 & B2 is under node X2 as shown below

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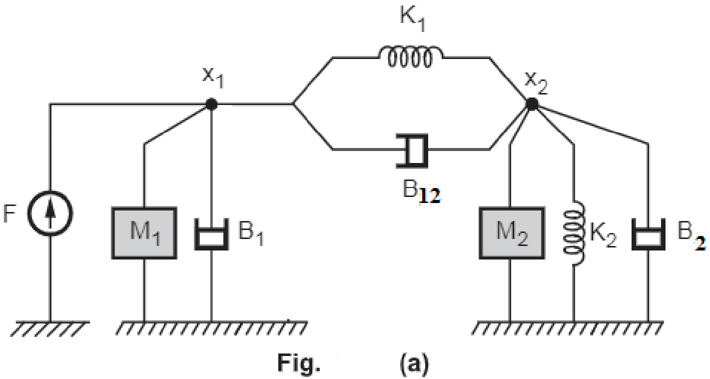


Figure(a) Equivalent mechanical network

$$\therefore M_1\frac{d^2x_1}{dt^2}+B_1\frac{dx_1}{dt}+B_{12}\frac{d}{dt}(x_1-x_2)+K_1(x_1-x_2)=f(t) \quad \text{.....(1)}$$

$$M_2\frac{d^2x_2}{dt^2}+B_2\frac{dx_2}{dt}+K_2x_2+B_{12}\frac{d}{dt}(x_2-x_1)+K_1(x_2-x_1)=0 \quad \text{.....(2)}$$

On replacing the displacements by velocity in the differential equations (1) and (2) system we get

$$\left(\text{i.e, } \frac{d^2x}{dt^2}=\frac{dv}{dt} \quad ; \quad \frac{dx}{dt}=v \text{ and } x=\int v\,dt\right)$$

$$M_1\frac{dv_1}{dt}+B_1v_1+B_{12}(v_1-v_2)+K_1\int(v_1-v_2)\,dt=f(t) \quad \text{.....(3)}$$

$$M_2\frac{dv_2}{dt}+B_2v_2+K_2\int v_2\,dt+B_{12}(v_2-v_1)+K_1\int(v_2-v_1)\,dt=0 \quad \text{.....(4)}$$

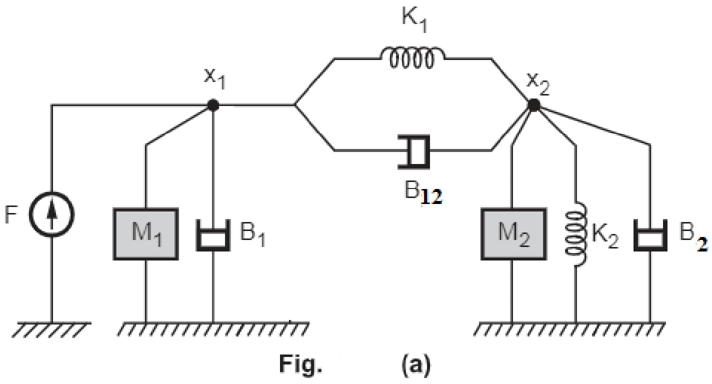
FORCE-VOLTAGE ANALOGOUS CIRCUIT

- The given mechanical system has two nodes (masses). Hence the force-voltage analogous electrical circuit will have two meshes.

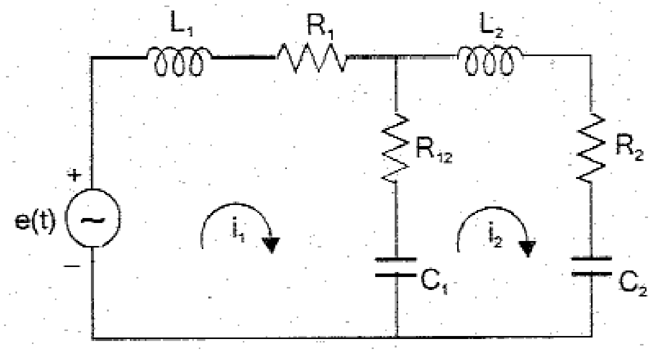
- The force applied to mass M1 is represented by a voltage source in first mesh. The elements M1 B1, K1 and B12 are connected to first node. Hence they are represented by analogous element in Mesh-1 forming a closed path.

- The elements K1, B12, M2, K2, and B2 are connected to second node. Hence they are represented by analogous element in Mesh-2 forming a closed path.

- The elements K1 and B12 are common between node-1 and 2 and so they are represented by analogous element as common elements between two meshes.

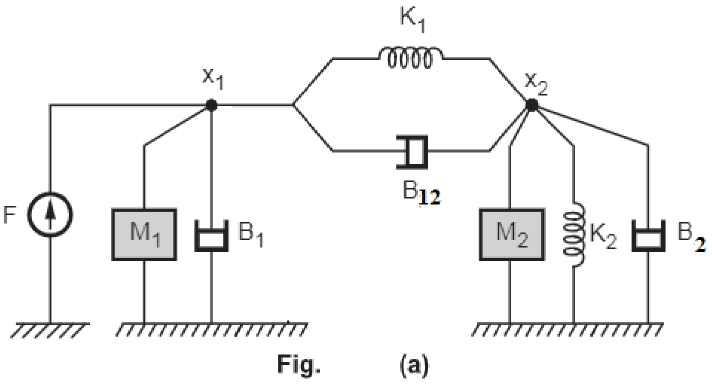


Figure(a) Equivalent mechanical network



Force-Voltage electrical analogous circuit

Analogous Quantities in FORCE-VOLTAGE system		
Mechanical System	Electrical System (Mesh Basis System)	
Force	Voltage e, v	**F(t)→e(t)**
Displacement, X	Charge, q	**X→q**
Velocity, V	Current, i	**V → i**
Dashpot, B	Resistance, R	**B →R**
Mass, M	Inductance, L	**M→L**
Spring, K	Inverse of Capacitance,1/C	**K→C**



Figure(a) Equivalent mechanical network

The Analogous Quantities in Force-Voltage system for the elements of mechanical system are

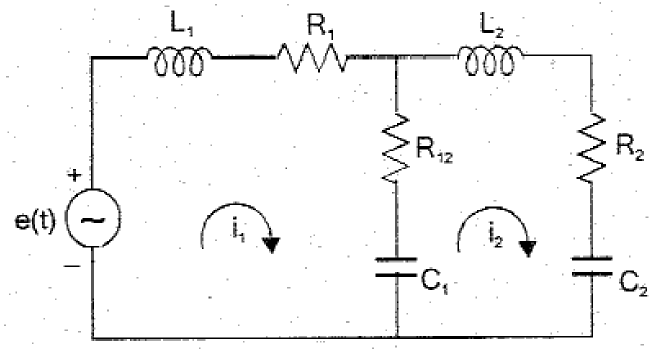
$F \to V$, $M \to L$, $B \to R$, $K \to 1/C$, $x \to q$,

$\dot{x} \to i$ (current), $x \to \int i\, dt$

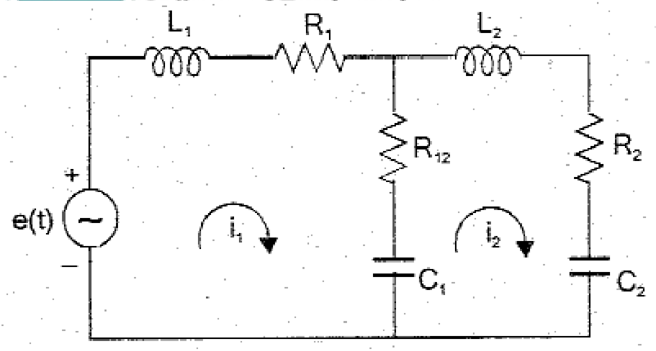
$f(t) \to e(t)$ $M_1 \to L_1$ $B_1 \to R_1$ $K_1 \to 1/C_1$

$v_1 \to i_1$ $M_2 \to L_2$ $B_2 \to R_2$ $K_2 \to 1/C_2$

$v_2 \to i_2$ $B_{12} \to R_{12}$



Force-voltage electrical analogous circuit



$$\therefore M_1\frac{d^2x_1}{dt^2}+B_1\frac{dx_1}{dt}+B_{12}\frac{d}{dt}(x_1-x_2)+K_1(x_1-x_2)=f(t) \quad ..(3)$$

$$M_2\frac{d^2x_2}{dt^2}+B_2\frac{dx_2}{dt}+K_2x_2+B_{12}\frac{d}{dt}(x_2-x_1)+K_1(x_2-x_1)=0 \quad .....(4)$$

Force-voltage electrical analogous circuit

The Analogous Quantities in Force-Voltage system for the elements of mechanical system are

- $f(t)\rightarrow e(t)$ $\quad M_1\rightarrow L_1$ $\quad B_1\rightarrow R_1$ $\quad K_1\rightarrow 1/C_1$

$v_1\rightarrow i_1$ $\quad M_2\rightarrow L_2$ $\quad B_2\rightarrow R_2$ $\quad K_2\rightarrow 1/C_2$

$v_2\rightarrow i_2$ $\quad B_{12}\rightarrow R_{12}$

By KVL for above circuit

$$L_1\frac{di_1}{dt}+R_1i_1+R_{12}(i_1-i_2)+\frac{1}{C_1}\int(i_1-i_2)dt=e(t) \quad .....(5)$$

$$L_2\frac{di_2}{dt}+R_2i_2+\frac{1}{C_2}\int i_2\,dt+R_{12}(i_2-i_1)+\frac{1}{C_1}\int(i_2-i_1)dt=0 \quad ...(6)$$

It is observed that the mesh basis equations (5) and (6) are similar to the differential equations (3) and (4) governing the mechanical system.

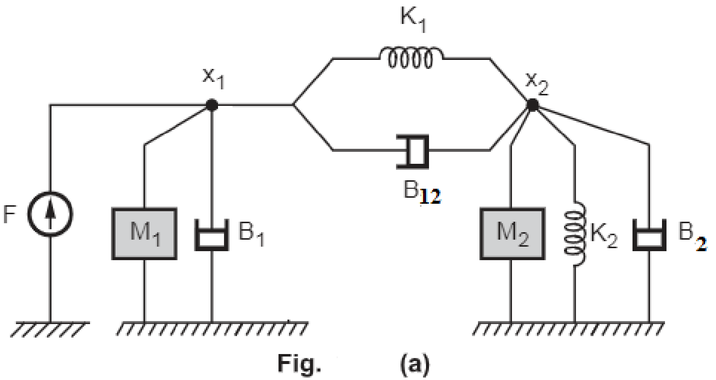
FORCE-CURRENT ANALOGOUS CIRCUIT

- The given mechanical system has two nodes (masses). Hence the force-current analogous electrical circuit will have two nodes.

- The force applied to mass M1 is represented by a Current source connected to Node-1 in analogous Electrical ckt. The elements M1 B1, K1 and B12 are connected to first node. Hence they are represented by analogous element in Node-1 in analogous Electrical ckt.

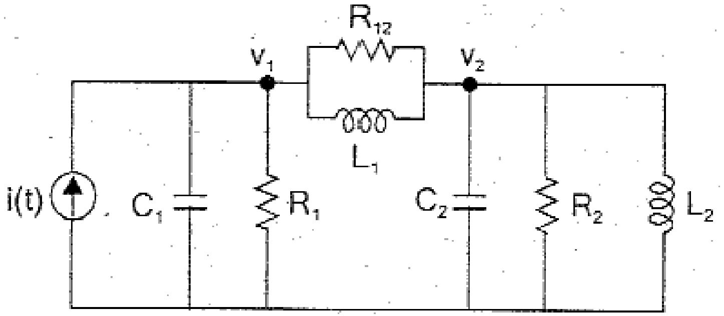
- The elements K1, B12, M2, K2, and B2 are connected to second node. Hence they are represented by analogous element as elements connected to Node-2 in analogous Electrical ckt.

- The elements K1 and B12 are common between node-1 and 2 and so they are represented by analogous element as common elements between two nodes in analogous Electrical ckt.



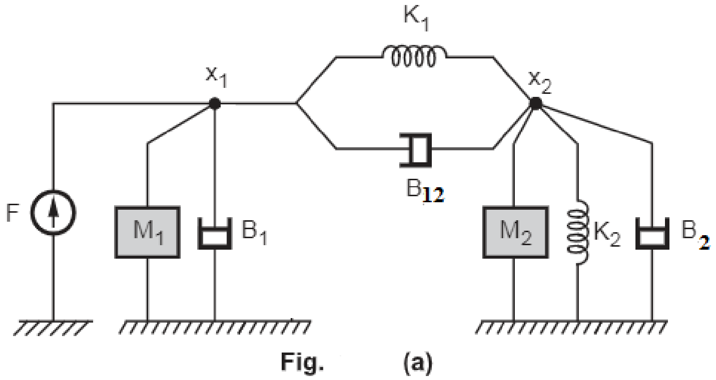
Figure(a) Equivalent mechanical network

$$F \rightarrow I,\quad M \rightarrow C,\quad B \rightarrow 1/R,\quad K \rightarrow 1/L,\quad x \rightarrow \phi,$$
$$\dot{x} = e(\text{e.m.f.}),\ x \rightarrow \int e\,dt$$



Force-Current electrical analogous circuit

Analogous Quantities in FORCE-CURRENT system		
Mechanical System	Electrical System (Mesh Basis System)	
Force	Current, i	**F(t)→i(t)**
Displacement, X	flux, Ø	**X→Ø**
Velocity, V	Voltage, v	**V → v**
Dashpot, B	Conductance G =1/R	**B →1/R**
Mass, M	Capacitance, C	**M→ C**
Spring, K	Inverse of Inductance,1/L	**K→1/L**



Figure(a) Equivalent mechanical network

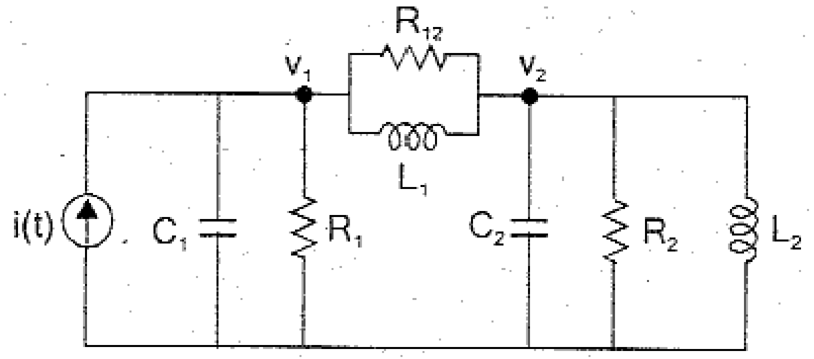
The Analogous Quantities in Force-Current system for the elements of mechanical system are

$F \to I, \quad M \to C, \quad B \to 1/R, \quad K \to 1/L, \quad x \to \phi,$
$\dot{x} = e(\text{e.m.f.}), \; x \to \int e \, dt$

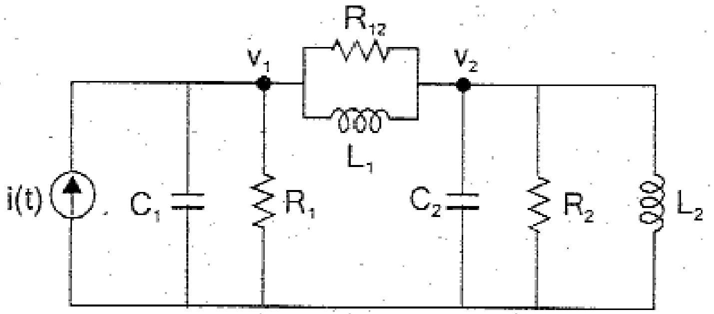
$f(t) \to i(t)$ $\quad M_1 \to C_1$ $\quad B_1 \to 1/R_1$ $\quad K_1 \to 1/L_1$

$v_1 \to v_1$ $\quad M_2 \to C_2$ $\quad B_2 \to 1/R_2$ $\quad K_2 \to 1/L_2$

$v_2 \to v_2$ $\quad B_{12} \to 1/R_{12}$



Force-Current electrical analogous circuit



$$\therefore M_1\frac{d^2x_1}{dt^2}+B_1\frac{dx_1}{dt}+B_{12}\frac{d}{dt}(x_1-x_2)+K_1(x_1-x_2)=f(t) \quad .....(3)$$

$$M_2\frac{d^2x_2}{dt^2}+B_2\frac{dx_2}{dt}+K_2x_2+B_{12}\frac{d}{dt}(x_2-x_1)+K_1(x_2-x_1)=0 \quad .....(4)$$

Force-Current electrical analogous circuit

The Analogous Quantities in Force-Current system for the elements of mechanical system are

- $F \to I,\quad M \to C,\quad B \to 1/R,\quad K \to 1/L,\quad x \to \phi,$
  $\dot{x} = e(\text{e.m.f.}),\ x \to \int e\, dt$

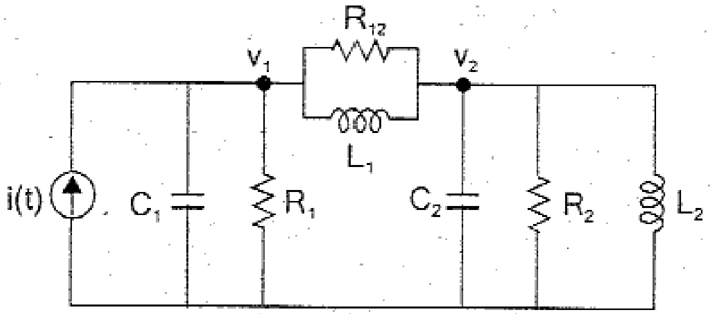
$f(t) \to i(t)$	$M_1 \to C_1$	$B_1 \to 1/R_1$	$K_1 \to 1/L_1$
$v_1 \to v_1$	$M_2 \to C_2$	$B_2 \to 1/R_2$	$K_2 \to 1/L_2$
$v_2 \to v_2$	$B_{12} \to 1/R_{12}$		

by KCL for above circuit

$$C_1\frac{dv_1}{dt}+\frac{1}{R_1}v_1+\frac{1}{R_{12}}(v_1-v_2)+\frac{1}{L_1}\int(v_1-v_2)dt=i(t) \quad .....(7)$$

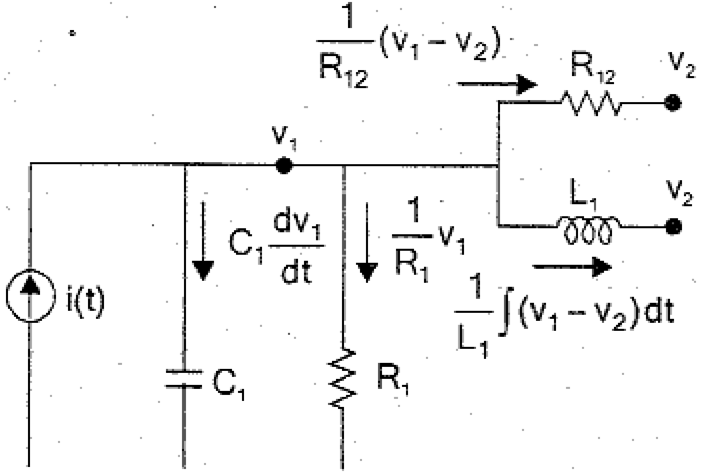
$$C_2\frac{dv_2}{dt}+\frac{1}{R_2}v_2+\frac{1}{L_2}\int v_2dt+\frac{1}{R_{12}}(v_2-v_1)+\frac{1}{L_1}\int(v_2-v_1)dt=0 \quad ...(8)$$

It is observed that the Node basis equations (7) and (8) are similar to the differential equations (3) and (4) governing the mechanical system.

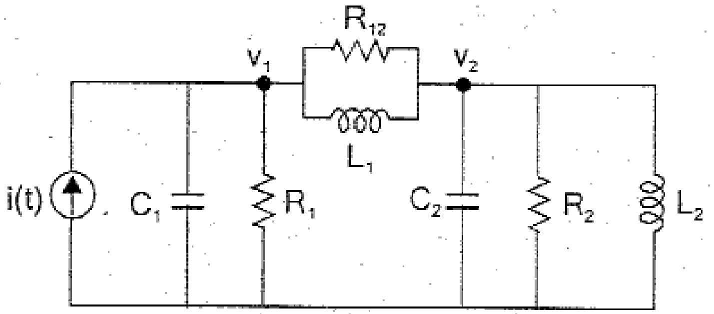


Force-Current electrical analogous circuit

The node basis equation by KCL for above circuit

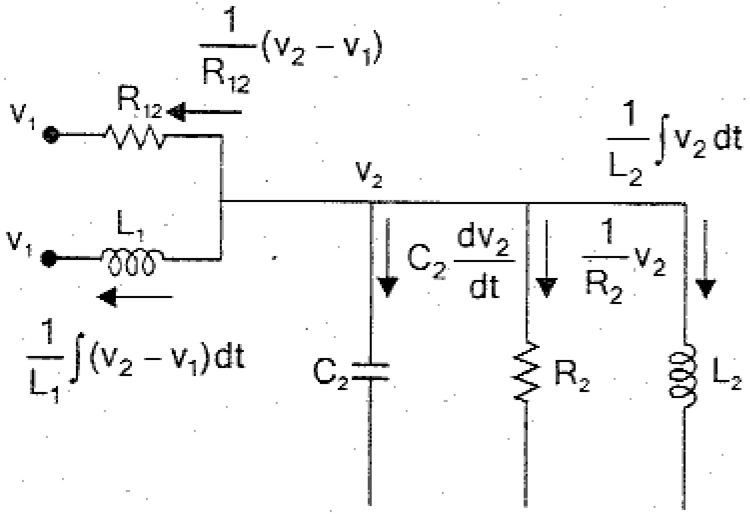


$$C_1\frac{dv_1}{dt} + \frac{1}{R_1}v_1 + \frac{1}{R_{12}}(v_1 - v_2) + \frac{1}{L_1}\int(v_1 - v_2)dt = i(t) \quad \ldots..(7)$$



Force-Current electrical analogous circuit

The node basis equation by KCL for above circuit



$$C_2\frac{dv_2}{dt} + \frac{1}{R_2}v_2 + \frac{1}{L_2}\int v_2 dt + \frac{1}{R_{12}}(v_2 - v_1) + \frac{1}{L_1}\int(v_2 - v_1)dt = 0 \quad \ldots..(8)$$

Write the differential equations governing the mechanical system shown in fig. Draw the force voltage and force-current electrical analogous circuits and verify by writing mesh and node equations.

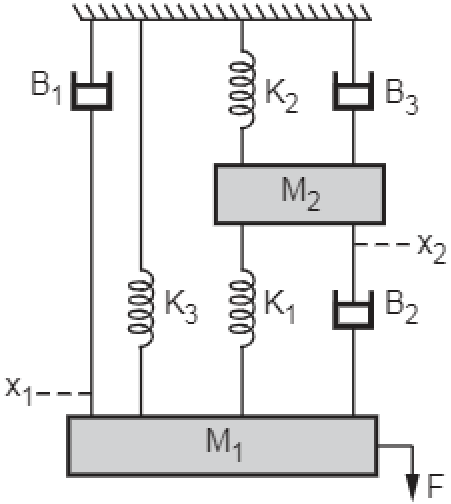


Figure. (1) Translational system

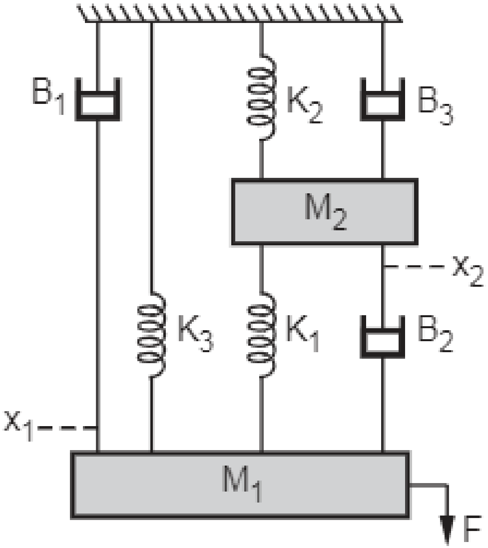
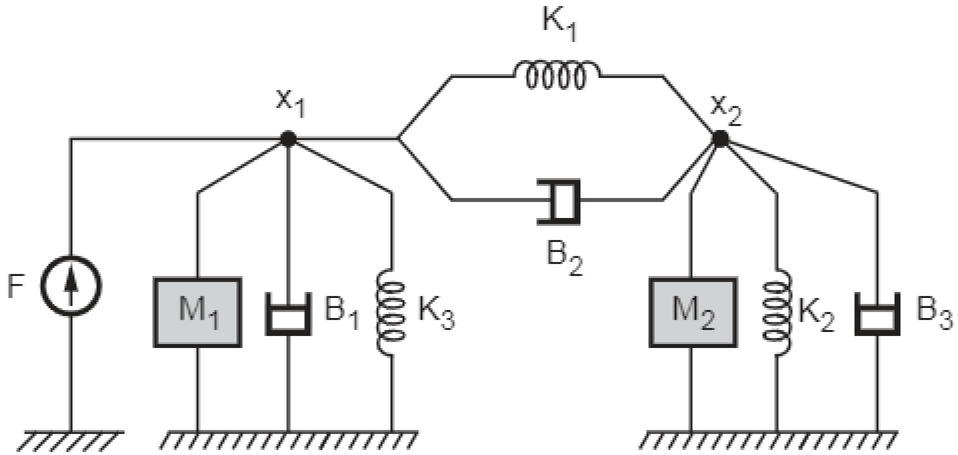


Figure. (1) Translational system



Figure(a) Equivalent mechanical network

$$F = M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + K_3x_1 + K_1(x_1 - x_2) + B_2 \frac{d(x_1 - x_2)}{dt} \quad \ldots(1)$$

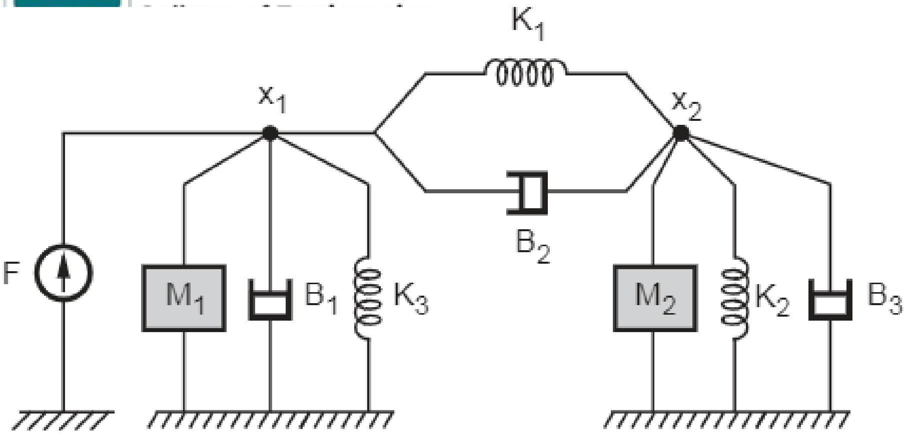
$$0 = K_1(x_2 - x_1) + B_2 \frac{d(x_1 - x_2)}{dt} + M_2 \frac{d^2x_2}{dt^2} + K_2x_2 + B_3 \frac{dx_2}{dt} \quad \ldots(2)$$

Analogous Quantities in FORCE-VOLTAGE system		
Mechanical System	Electrical System (Mesh Basis System)	
Force	Voltage e, v	**F(t)→e(t)**
Displacement, X	Charge, q	**X→q**
Velocity, V	Current, i	**V → i**
Dashpot, B	Resistance, R	**B →R**
Mass, M	Inductance, L	**M→L**
Spring, K	Inverse of Capacitance,1/C	**K→C**

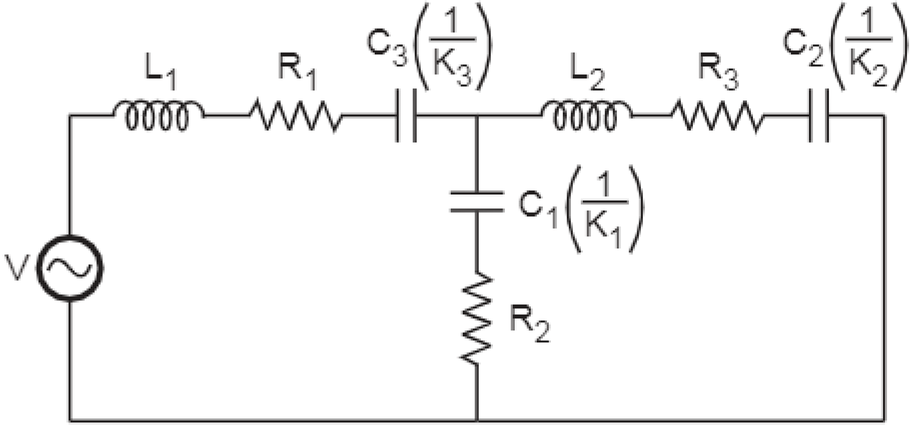
The Analogous Quantities in Force-Voltage system for the elements of mechanical system are

$$\textbf{F-V analogy}: M \to L,\ B \to R,\ K \to \frac{1}{C},\ \frac{dx}{dt} \to i,\quad x \to \int i\, dt,\ \frac{d^2x}{dt^2} \to \frac{di}{dt}$$

$$F \to V,\quad M \to L,\ B \to R,\ K \to 1/C,\ x \to q,$$
$$\dot{x} \to i\ (\text{current}),\ x \to \int i\, dt$$



Figure(a) Equivalent mechanical network



Force-voltage electrical analogous circuit

F-V analogy : $M \to L,\ B \to R,\ K \to \frac{1}{C},\ \frac{dx}{dt} \to i,\ \ x \to \int i\,dt,\ \frac{d^2x}{dt^2} \to \frac{di}{dt}$

$$\therefore V = L_1 \frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_3}\int i_1 dt\ \frac{1}{C_1}\int (i_1 - i_2)\,dt + R_2\ (i_1 - i_2) \qquad \ldots(3)$$

$$\therefore\ 0 = \frac{1}{C_1}\int (i_2 - i_1)\,dt + R_2\ (i_2 - i_1) + L_2 \frac{di_2}{dt} + \frac{1}{C_2}\int i_2 dt + R_3\ i_2 \qquad \ldots(4)$$

Write the differential equations governing the mechanical system shown in fig. Draw the force voltage and force-current electrical analogous circuits and verify by writing mesh and node equations.

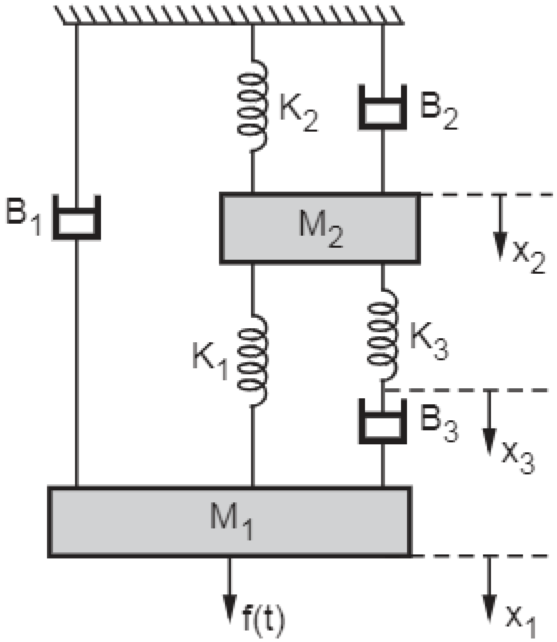
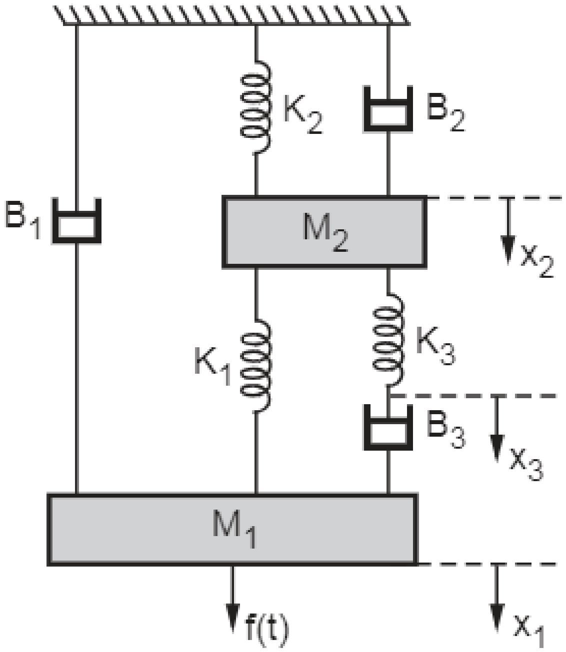
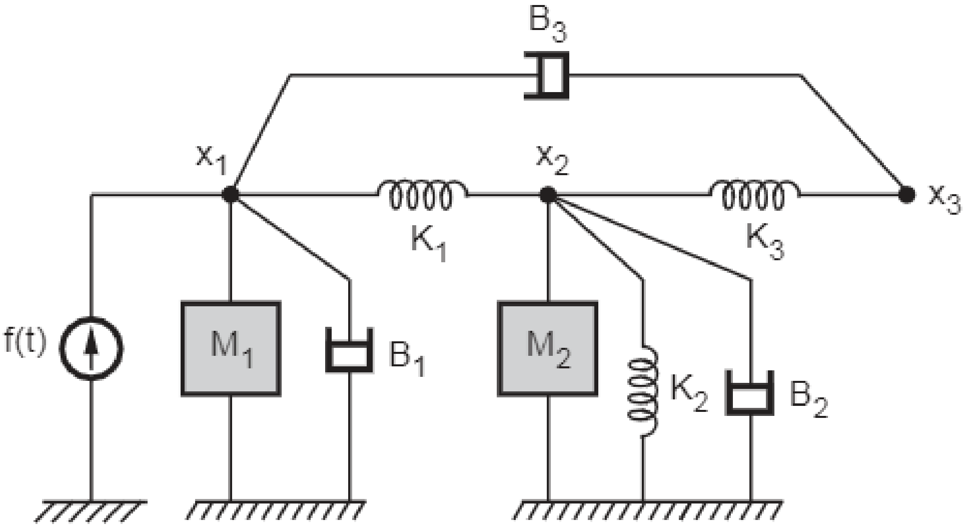


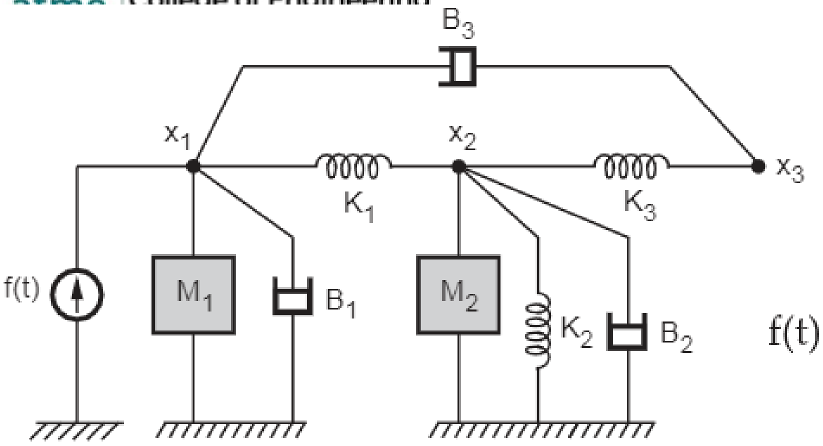
Figure. (1) Translational system

Sol. : M_1 and B_1 are under the displacement x_1. K_1 is between x_1 and x_2 while B_3 is between x_1 and x_3. K_3 is between x_3 and x_2. The mass M_2, K_2 and B_2 are under x_2. Hence the equivalent mechanical system is as shown in the Fig. (a).





Figure(a) Equivalent mechanical network

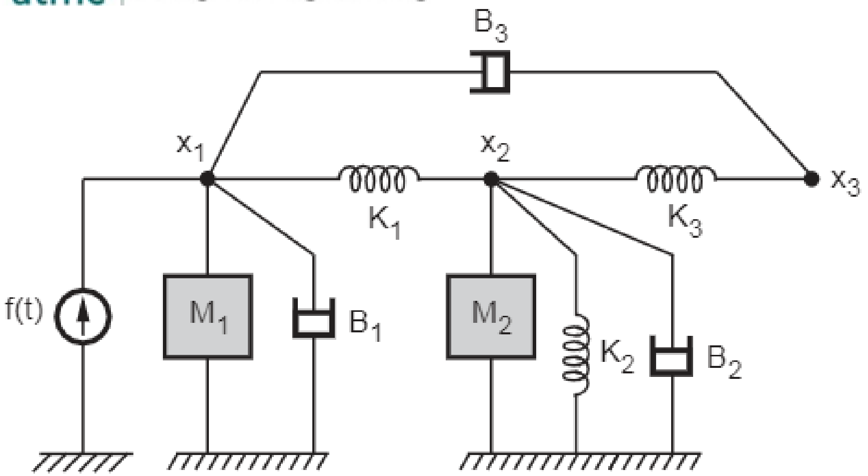


Figure(a) Equivalent mechanical network

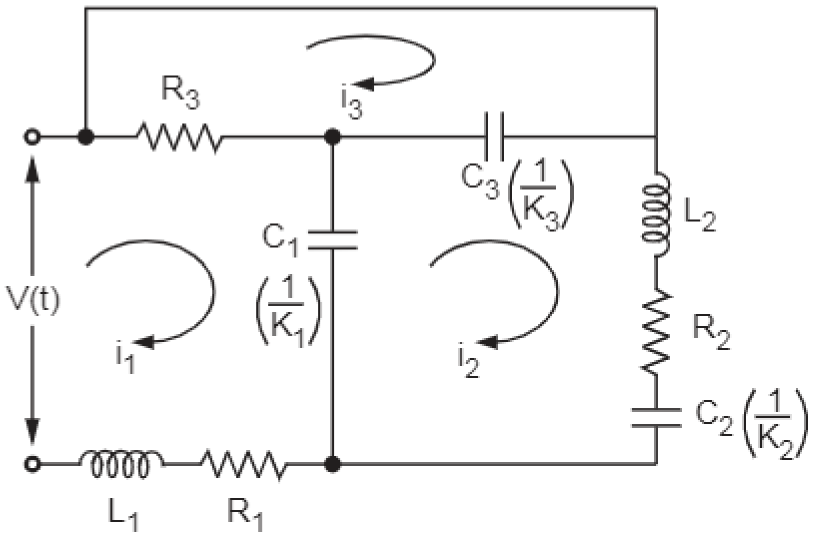
$$f(t) = M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + K_1 (x_1 - x_2) + B_3 \frac{d(x_1 - x_3)}{dt} \quad \ldots(1)$$

$$0 = K_1 (x_2 - x_1) + M_2 \frac{d^2x_2}{dt^2} + B_2 \frac{dx_2}{dt} + K_2 x_2 + K_3 (x_2 - x_3) \quad (2)$$

$$0 = K_3 (x_3 - x_2) + B_3 \frac{d(x_3 - x_1)}{dt} \quad \ldots(3)$$

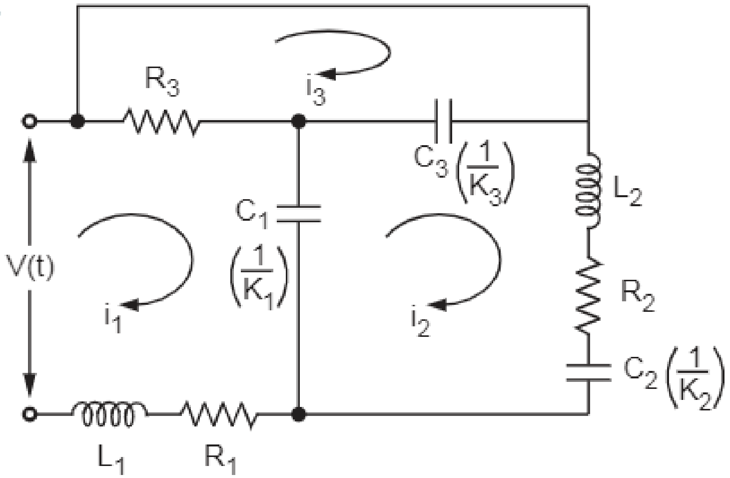


Figure(a) Equivalent mechanical network



Force-voltage electrical analogous circuit

F-V analogy : $M \rightarrow L,\ B \rightarrow R,\ K \rightarrow \frac{1}{C},\ \frac{dx}{dt} \rightarrow i,\ \ x \rightarrow \int i\,dt,\ \frac{d^2x}{dt^2} \rightarrow \frac{di}{dt}$



Force-voltage electrical analogous circuit

$$f(t) = M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + K_1 (x_1 - x_2) + B_3 \frac{d(x_1 - x_3)}{dt} \qquad \ldots(1)$$

$$0 = K_1 (x_2 - x_1) + M_2 \frac{d^2x_2}{dt^2} + B_2 \frac{dx_2}{dt} + K_2 x_2 + K_3 (x_2 - x_3) \quad ..\ (2)$$

$$0 = K_3 (x_3 - x_2) + B_3 \frac{d(x_3 - x_1)}{dt} \qquad \ldots(3)$$

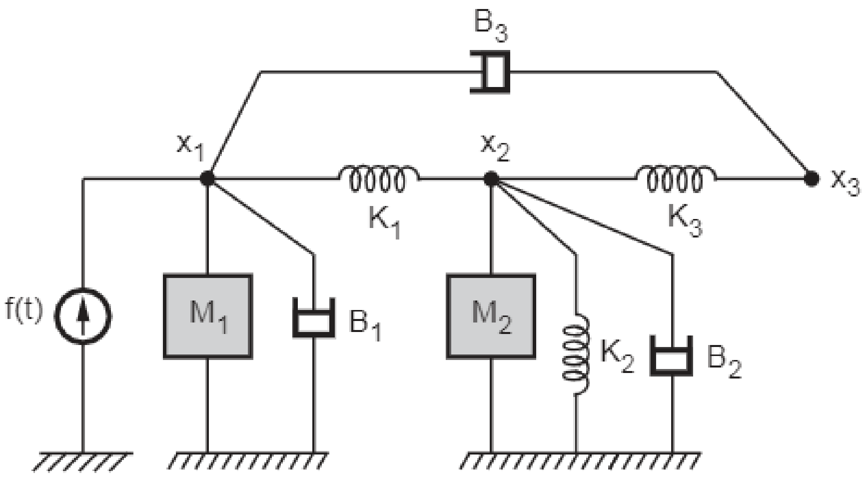
F-V analogy : $M \rightarrow L,\ B \rightarrow R,\ K \rightarrow \frac{1}{C},\ \frac{dx}{dt} \rightarrow i,\ x \rightarrow \int i\,dt,\ \frac{d^2x}{dt^2} \rightarrow \frac{di}{dt}$

By KVL for above circuit

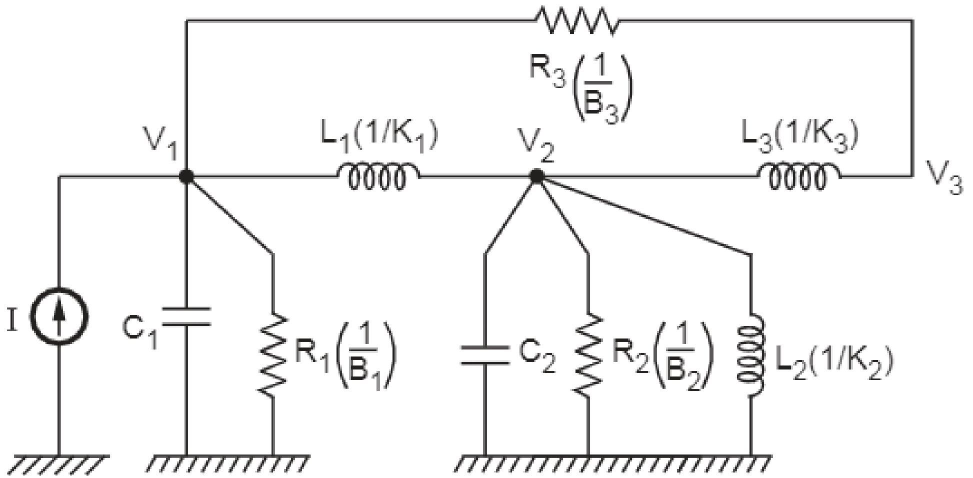
$$V(t) = L_1 \frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_1} \int (i_1 - i_2)\,dt + R_3 (i_1 - i_3) \qquad ..\ (4)$$

$$0 = \frac{1}{C_1} \int (i_2 - i_1)\,dt + L_2 \frac{di_2}{dt} + R_2 i_2 + \frac{1}{C_2} \int i_2\,dt + \frac{1}{C_3} \int (i_2 - i_3)\,dt \qquad ..\ (5)$$

$$0 = \frac{1}{C_3} \int (i_3 - i_2)\,dt + R_3 (i_3 - i_1) \qquad ..\ (6)$$

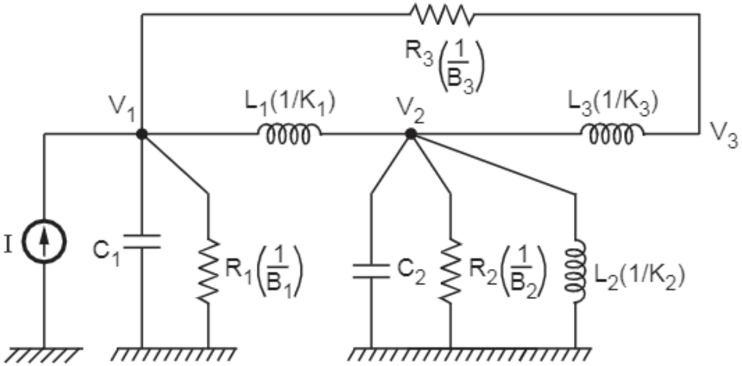


Figure(a) Equivalent mechanical network



Force-Current electrical analogous circuit

$$\textbf{F-I analogy :}\ M \rightarrow C,\ B \rightarrow \frac{1}{R},\ K \rightarrow \frac{1}{L},\ x \rightarrow \phi,\ \frac{dx}{dt} \rightarrow v,\quad x \rightarrow \int v\,dt,\ \frac{d^2x}{dt^2} \rightarrow \frac{dv}{dt}$$



$$f(t) = M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + K_1 (x_1 - x_2) + B_3 \frac{d(x_1 - x_3)}{dt} \quad \ldots(1)$$

$$0 = K_1 (x_2 - x_1) + M_2 \frac{d^2x_2}{dt^2} + B_2 \frac{dx_2}{dt} + K_2 x_2 + K_3 (x_2 - x_3) \quad .. (2)$$

$$0 = K_3 (x_3 - x_2) + B_3 \frac{d(x_3 - x_1)}{dt} \quad \ldots(3)$$

Force-Current electrical analogous circuit

F-I analogy : $M \rightarrow C,\ B \rightarrow \frac{1}{R},\ K \rightarrow \frac{1}{L},\ x \rightarrow \phi,\ \frac{dx}{dt} \rightarrow v,\ x \rightarrow \int v\,dt,\ \frac{d^2x}{dt^2} \rightarrow \frac{dv}{dt}$

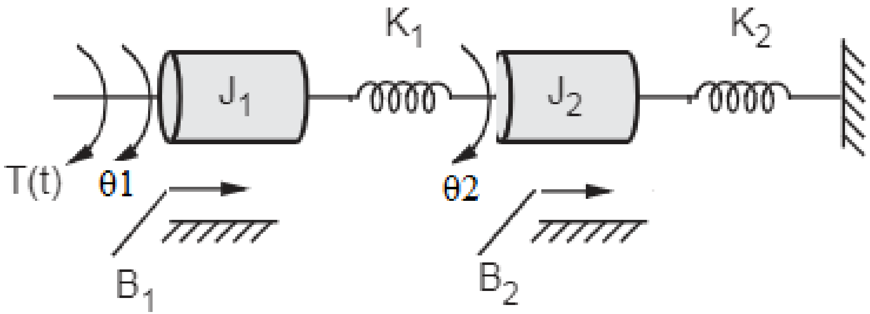
By KCL for above circuit

$$I(t) = C_1 \frac{dv_1}{dt} + \frac{1}{R_1} v_1 + \frac{1}{L_1} \int (v_1 - v_2)\,dt + \frac{1}{R_3} (v_1 - v_3) \quad .. (4)$$

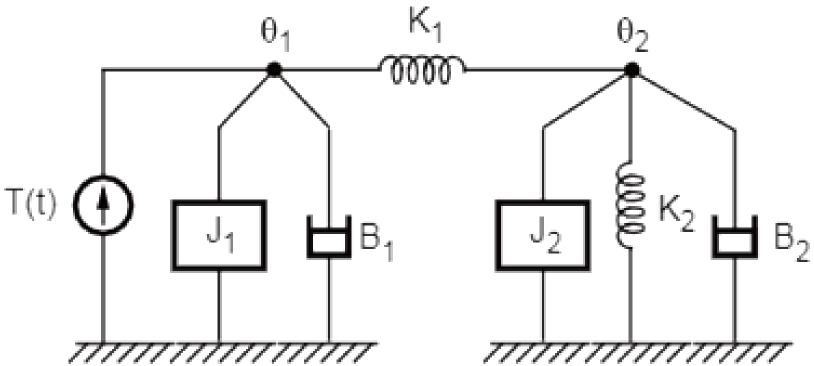
$$0 = \frac{1}{L_1} \int (v_2 - v_1)dt + C_2 \frac{dv_2}{dt} + \frac{1}{R_2} v_2 + \frac{1}{L_2} \int v_2 dt + \frac{1}{L_3} \int (v_2 - v_3)dt \quad .. (5)$$

$$0 = \frac{1}{L_3} \int (v_3 - v_2)\,dt + \frac{1}{R_3} (v_3 - v_1) \quad .. (6)$$

Write the differential equations governing the mechanical rotational system shown in fig. Draw the Torque-current electrical analogy give all performance equation



- J1 and B1 are under Ө1.
- Spring K1 is between Ө1 and Ө2.
- J2, B2 and K2 are under Ө2.



The equivalent mechanical system

$$T(t) = J_1 \frac{d^2\theta_1}{dt^2} + B_1 \frac{d\theta_1}{dt} + K_1(\theta_1 - \theta_2) \qquad \ldots 1$$

$$0 = K_1(\theta_2 - \theta_1) + J_2 \frac{d^2\theta_2}{dt^2} + B_2 \frac{d\theta_2}{dt} + K_2\theta_2 \qquad \ldots 2$$

Electrical Analogous of Mechanical systems

Mechanical System		Electrical System	
Translational Mechanical System	**Rotational Mechanical System**	**Force/Torque-Voltage Analogy**	**Force/Torque-Current Analogy**
Force(F)	Torque(T)	Voltage(V)	Current(i)
Mass(M)	Moment of inertia(J)	Inductance(L)	Capacitance(C)
Frictional coefficient(B)	Rotational friction coefficient(B)	Resistance(R)	Reciprocal of Resistance(1/R)
Spring constant(K)	Torsional spring constant(K)	Reciprocal of Capacitance (1/c)	Reciprocal of Inductance(1/L)
Displacement(x)	Angular displacement(θ)	Charge(q)	Magnetic flux(Ø)
Velocity(v)	Angular velocity(ω)	Current(i)	Voltage(V)

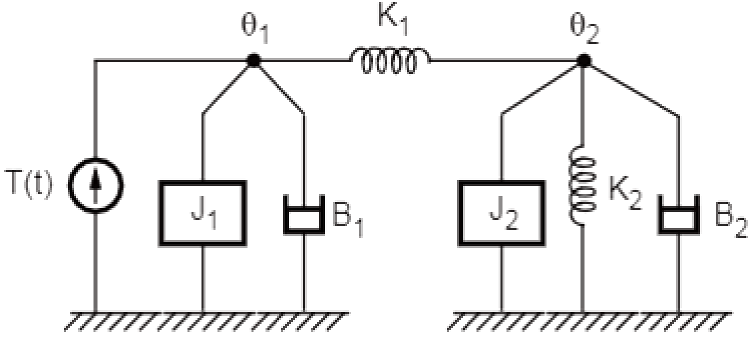
ANALOGOUS QUANTITIES OF FORCE-CURRENT ANALOGY & TORQUE-CURRENT ANALOGY

Translational	Rotational	Electrical
F Force	T	Current I
M Mass	J	C
B friction	B	1/R
K Spring	K	1/L
x displacement	θ	ϕ
$\dot{x}$ Velocity $= \frac{dx}{dt}$	$\dot{\theta} = \frac{d\theta}{dt} = \omega$	Voltage 'v' $= \frac{d\phi}{dt}$

For T - I analogy,

$$J \to C,\ B \to \frac{1}{R},\ K \to \frac{1}{L},\ \theta \to \phi,\ \frac{d\theta}{dt} \to V,\ \theta \to \int V dt$$

ANALOGOUS QUANTITIES OF TORQUE-CURRENT ANALOGY



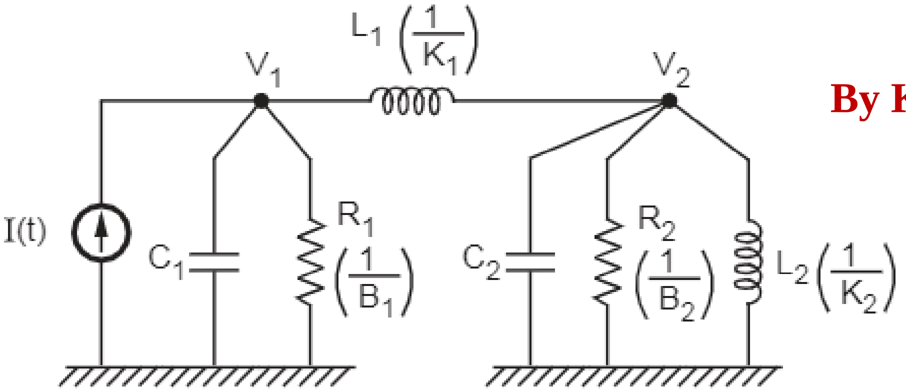
The equivalent mechanical system

$$T(t) = J_1 \frac{d^2\theta_1}{dt^2} + B_1 \frac{d\theta_1}{dt} + K_1(\theta_1 - \theta_2) \quad \ldots 1$$

$$0 = K_1(\theta_2 - \theta_1) + J_2 \frac{d^2\theta_2}{dt^2} + B_2 \frac{d\theta_2}{dt} + K_2\theta_2 \quad \ldots 2$$

For T - I analogy,

$$J \rightarrow C,\ B \rightarrow \frac{1}{R},\ K \rightarrow \frac{1}{L},\ \theta \rightarrow \phi,\ \frac{d\theta}{dt} \rightarrow V,\ \theta \rightarrow \int V dt$$



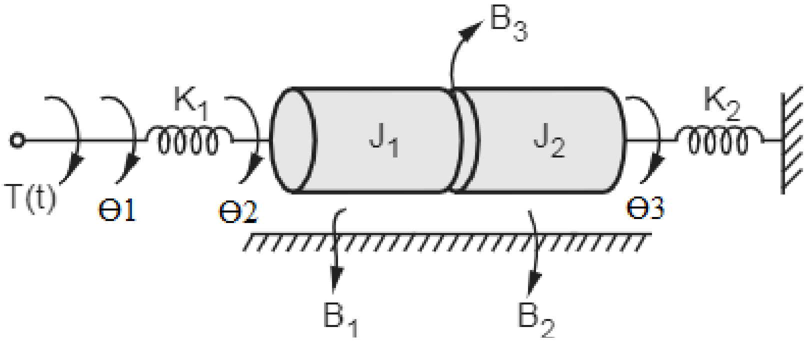
Torque-Current electrical analogous circuit

By KCL for Torque-Current electrical analogous circuit

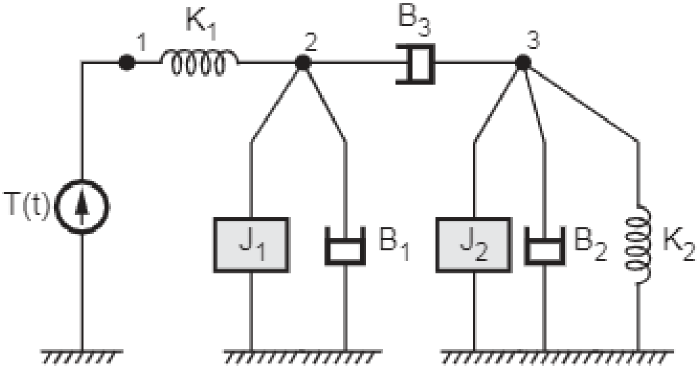
$$I(t) = C_1 \frac{dV_1}{dt} + \frac{1}{R_1} V_1 + \frac{1}{L_1} \int (V_1 - V_2) dt \quad \ldots 3$$

$$0 = \frac{1}{L_1} \int (V_2 - V_1)\, dt + C_2 \frac{dV_2}{dt} + \frac{1}{R_2} V_2 + \frac{1}{L_2} \int V_2\, dt \quad \ldots 4$$

Write the differential equations governing the mechanical rotational system shown in fig. Draw the Torque-current electrical analogy give all performance equation



- Spring K1 is between Θ1 and Θ2.
- J1 and B1 are under displacement Θ2.
- B3 is between Θ2 and Θ3
- J2, B2 and K2 are under displacement Θ3.



The equivalent mechanical system

$$T(t) = K_1(\theta_1 - \theta_2) \qquad \ldots 1$$

$$0 = K_1(\theta_2 - \theta_1) + J_1\frac{d^2\theta_2}{dt^2} + B_1\frac{d\theta_2}{dt} + \frac{B_3 d(\theta_2 - \theta_3)}{dt} \qquad \ldots 2$$

$$0 = B_3\frac{d(\theta_3 - \theta_2)}{dt} + J_2\frac{d^2\theta_3}{dt^2} + B_2\frac{d\theta_3}{dt} + K_2\theta_3 \qquad \ldots 3$$

Electrical Analogous of Mechanical systems

Mechanical System		Electrical System	
Translational Mechanical System	**Rotational Mechanical System**	**Force/Torque-Voltage Analogy**	**Force/Torque-Current Analogy**
Force(F)	Torque(T)	Voltage(V)	Current(i)
Mass(M)	Moment of inertia(J)	Inductance(L)	Capacitance(C)
Frictional coefficient(B)	Rotational friction coefficient(B)	Resistance(R)	Reciprocal of Resistance(1/R)
Spring constant(K)	Torsional spring constant(K)	Reciprocal of Capacitance (1/c)	Reciprocal of Inductance(1/L)
Displacement(x)	Angular displacement(θ)	Charge(q)	Magnetic flux(Ø)
Velocity(v)	Angular velocity(ω)	Current(i)	Voltage(V)

ANALOGOUS QUANTITIES OF FORCE-CURRENT ANALOGY & TORQUE-CURRENT ANALOGY

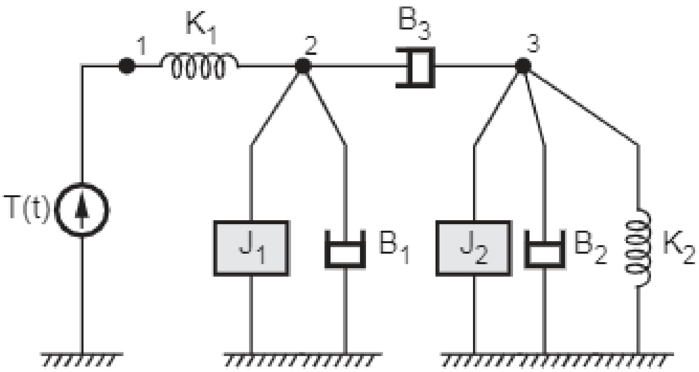
Translational	Rotational	Electrical
F Force	T	Current I
M Mass	J	C
B friction	B	1/R
K Spring	K	1/L
x displacement	θ	ϕ
$\dot{x}$ Velocity $= \frac{dx}{dt}$	$\dot{\theta} = \frac{d\theta}{dt} = \omega$	Voltage 'v' $= \frac{d\phi}{dt}$

For T - I analogy,

$$J \to C,\ B \to \frac{1}{R},\ K \to \frac{1}{L},\ \theta \to \phi,\ \frac{d\theta}{dt} \to V,\ \theta \to \int V dt$$

$\theta \to \int v(t)\ dt, \frac{d^2\theta}{dt^2} \to \frac{dv\ (t)}{dt}$ and use in equations (1), (2) and (3).

ANALOGOUS QUANTITIES OF TORQUE-CURRENT ANALOGY

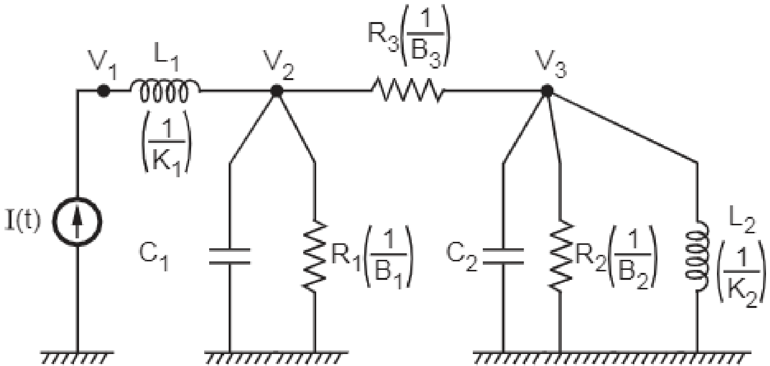


The equivalent mechanical system

$$T(t) = K_1(\theta_1-\theta_2) \qquad \ldots 1$$

$$0 = K_1(\theta_2-\theta_1)+J_1\frac{d^2\theta_2}{dt^2}+B_1\frac{d\theta_2}{dt}+\frac{B_3 d(\theta_2-\theta_3)}{dt} \qquad \ldots 2$$

$$0 = B_3\frac{d(\theta_3-\theta_2)}{dt}+J_2\frac{d^2\theta_3}{dt^2}+B_2\frac{d\theta_3}{dt}+K_2\theta_3 \qquad \ldots 3$$



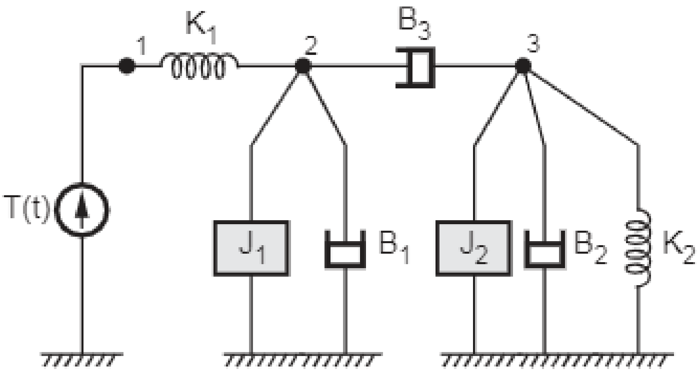
For T - I analogy,

$$J \rightarrow C,\ B \rightarrow \frac{1}{R},\ K \rightarrow \frac{1}{L},\ \theta \rightarrow \phi,\ \frac{d\theta}{dt} \rightarrow V,\ \theta \rightarrow \int V dt$$

$\theta \rightarrow \int v(t)\ dt, \frac{d^2\theta}{dt^2} \rightarrow \frac{dv(t)}{dt}$ and use in equations (1), (2) and (3).

Torque-Current electrical analogous circuit

ANALOGOUS QUANTITIES OF TORQUE-CURRENT ANALOGY

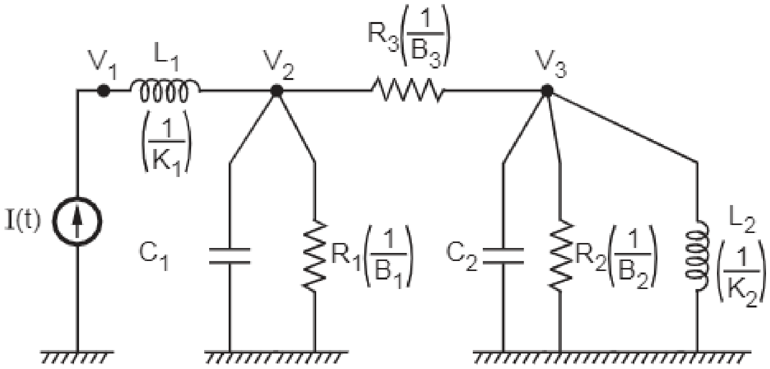


The equivalent mechanical system

$$T(t) = K_1(\theta_1 - \theta_2) \quad \ldots 1$$

$$0 = K_1(\theta_2 - \theta_1) + J_1 \frac{d^2\theta_2}{dt^2} + B_1 \frac{d\theta_2}{dt} + \frac{B_3 d(\theta_2 - \theta_3)}{dt} \quad \ldots 2$$

$$0 = B_3 \frac{d(\theta_3 - \theta_2)}{dt} + J_2 \frac{d^2\theta_3}{dt^2} + B_2 \frac{d\theta_3}{dt} + K_2 \theta_3 \quad \ldots 3$$



Torque-Current electrical analogous circuit

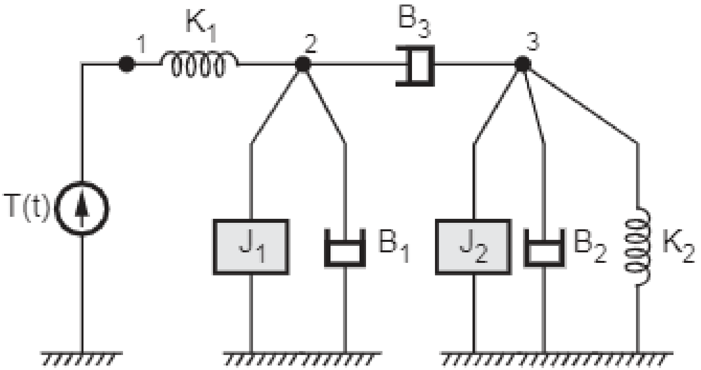
By KCL for Torque-Current electrical analogous circuit

$$I(t) = \frac{1}{L_1} \int (V_1 - V_2)\, dt$$

$$0 = \frac{1}{L_1} \int (V_2 - V_1)\, dt + C_1 \frac{dV_2}{dt} + \frac{1}{R_1} V_2 + \frac{1}{R_3}(V_2 - V_3)$$

$$0 = \frac{1}{R_3}(V_3 - V_2) + C_2 \frac{dV_3}{dt} + \frac{1}{R_2} V_3 + \frac{1}{L_2} \int V_3\, dt$$

ANALOGOUS QUANTITIES OF TORQUE-VOLTAGE ANALOGY



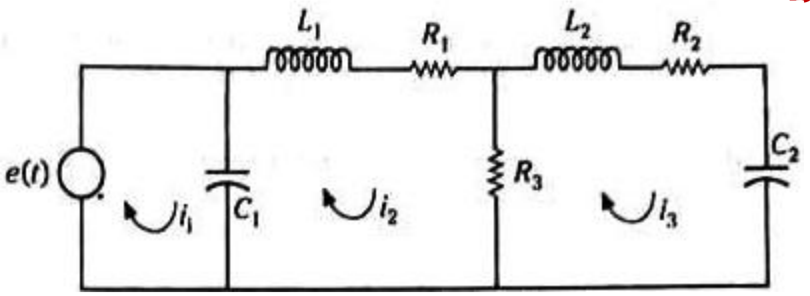
The equivalent mechanical system

$$T(t) = K_1(\theta_1 - \theta_2) \qquad \ldots 1$$

$$0 = K_1(\theta_2-\theta_1) + J_1 \frac{d^2\theta_2}{dt^2} + B_1 \frac{d\theta_2}{dt} + \frac{B_3 d(\theta_2-\theta_3)}{dt} \qquad \ldots 2$$

$$0 = B_3 \frac{d(\theta_3-\theta_2)}{dt} + J_2 \frac{d^2\theta_3}{dt^2} + B_2 \frac{d\theta_3}{dt} + K_2\theta_3 \qquad \ldots 3$$

By KCL for Torque-Voltage electrical analogous circuit



$$e(t) = \frac{1}{C_1}\int (i_1 - i_2)\,dt$$

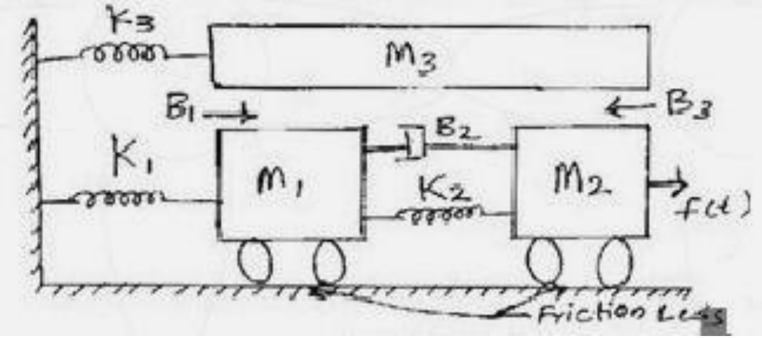
$$L_1 \frac{di_2}{dt} + R_1 i_2 + R_3(i_2 - i_3) + \frac{1}{C_1}\int (i_1 - i_2)\,dt = 0$$

$$L_2 \frac{di_3}{dt} + R_2 i_3 + R_3(i_3 - i_2) + \frac{1}{C_2}\int i_3\,dt = 0$$

Torque-Voltage electrical analogous circuit

Assignment Question

Write the differential equations governing the mechanical Translational system shown in fig. Draw the Force-Voltage electrical analogy give all performance equation



- (M1, K1)is under the influence of Displacement X1.
- M2 under displacement X2.
- (M3, K3)is under the influence of Displacement X3.
- B1 is under influence of both displacement (X1 & X3).
- B2, K2 is under influence of both displacement (X1 & X2)
- B3 is under influence of both displacement (X2 & X3)

Laplace transform method

- The Laplace transform method is an operational method that can be used advantageously for solving linear differential equations. By use of Laplace transforms, we can convert many common functions, such as sinusoidal functions, damped sinusoidal functions, and exponential functions, into algebraic functions of a complex variables. Operations such as differentiation and integration can be replaced by algebraic operations in the complex plane. Thus, a linear differential equation can be transformed into an algebraic equation in a complex variables. If the algebraic equation in s is solved for the dependent variable, then the solution of the differential equation (the inverse Laplace transform of the dependent variable) may be found by use of a Laplace transform table or by use of the partial-fraction expansion technique.

- An advantage of the Laplace transform method is that it allows the use of graphical techniques for predicting the system performance without actually solving system differential equations. Another advantage of the Laplace transform method is that, when we solve the differential equation, both the transient component and steady- state component of the solution can be obtained simultaneously.

Why it is important to understand: Transients and Laplace transforms

- A transient state will exist in a circuit containing one or more energy storage elements (i.e. capacitors and inductors) whenever the energy conditions in the circuit change, until the new steady state condition is reached.

- Transients are caused by changing the applied voltage or current, or by changing any of the circuit elements; such changes occur due to opening and closing switches.

- Equations are developed analytically by using both differential equations and Laplace transforms for different waveform supply voltages.

- The solution of most electrical problems can be reduced ultimately to the solution of differential equations and the use of Laplace transforms **to solve differential equations** provides a convenient method for the calculation of the complete response of a circuit. I.e are used to analyze transient responses directly from circuit diagrams

The Laplace transform is a particularly elegant way to solve linear differential equations with constant coefficients. The Laplace transform describes signals and systems not as functions of time, but as functions of a complex variable

When transformed into the Laplace domain, differential equations become polynomials of

Solving a differential equation in the time domain becomes a simple polynomial multiplication and division in the Laplace domain. However, the input and output signals are also in the Laplace domain, and any system response must undergo an inverse Laplace transform to become a meaningful time-dependent signal.

Table Standard Laplace transforms

Table 45.1 Standard Laplace transforms

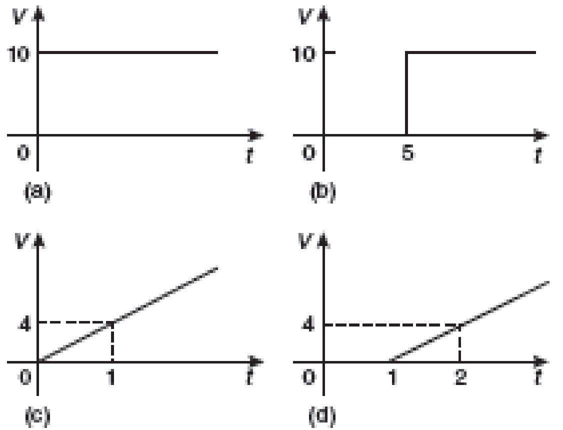
Time function $f(t)$	Laplace transform $\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$
1. δ (unit impulse)	1
2. 1 (unit step function)	$\frac{1}{s}$
3. e^{at} (exponential function)	$\frac{1}{s-a}$
4. unit step delayed by T	$\frac{e^{-sT}}{s}$
5. $\sin \omega t$ (sine wave)	$\frac{\omega}{s^2+\omega^2}$
6. $\cos \omega t$ (cosine wave)	$\frac{s}{s^2+\omega^2}$
7. t (unit ramp function)	$\frac{1}{s^2}$
8. t^2	$\frac{2!}{s^3}$
9. t^n $(n=1, 2, 3\ldots)$	$\frac{n!}{s^{n+1}}$
	s
10. $\cosh \omega t$	$\frac{s}{s^2-\omega^2}$
11. $\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}$
12. $e^{at} t^n$	$\frac{n!}{(s-a)^{n+1}}$
13. $e^{-at} \sin \omega t$ (damped sine wave)	$\frac{\omega}{(s+a)^2+\omega^2}$
14. $e^{-at} \cos \omega t$ (damped cosine wave)	$\frac{s+a}{(s+a)^2+\omega^2}$
15. $e^{-at} \sinh \omega t$	$\frac{\omega}{(s+a)^2-\omega^2}$
16. $e^{-at} \cosh \omega t$	$\frac{s+a}{(s+a)^2-\omega^2}$

(a) a step voltage of 10 V which starts at time $t = 0$

(b) a step voltage of 10 V which starts at time $t = 5$ s

(c) a ramp voltage which starts at zero and increases at 4 V/s

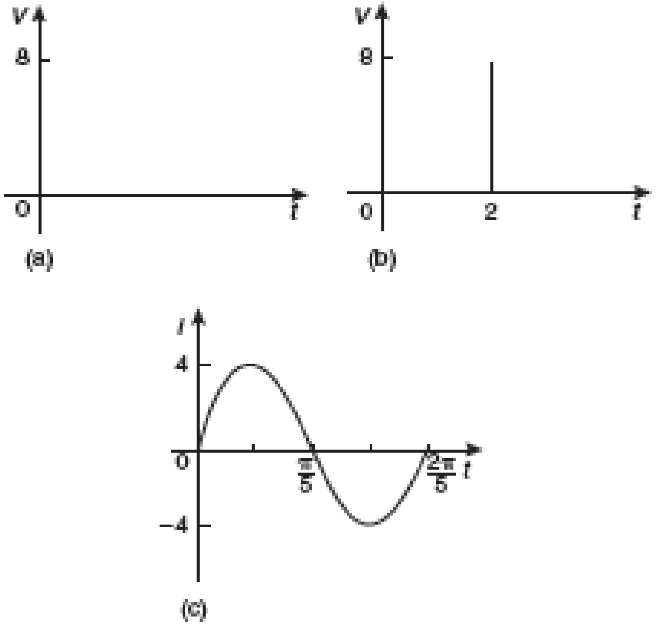
(d) a ramp voltage which starts at time $t = 1$ s and increases at 4 V/s



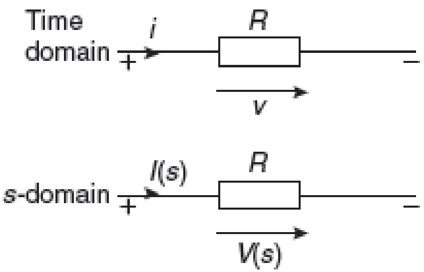
an impulse voltage of 8 V which starts at time $t = 0$

an impulse voltage of 8 V which starts at time $t = 2$ s

a sinusoidal current of 4 A and angular frequency 5 rad/s which starts at time $t = 0$

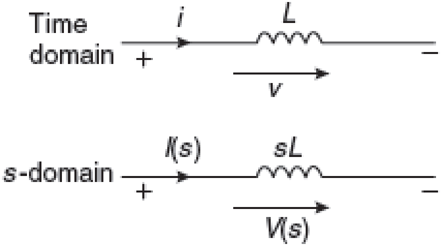


The Resistor, Inductor & Capacitor is shown in both the time domain and the *s*-domain.



Since v and i are both functions of time, a more correct equation would be v(t)=R*i(t)

The Laplace transform of this equation is: V(s) = R*I(s)



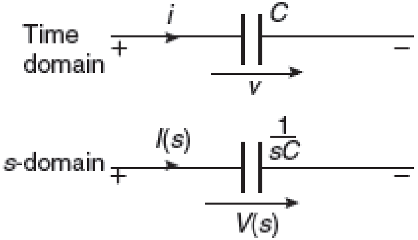
$$V = L*(di/dt)$$

The Laplace transform of the equation is:

$$V(s)=sL*I(s)$$

Impedance of the inductor in the *s*-domain is given by:

$$Z(s) = V(s)/I(s)= \mathbf{sL}$$



$$i =C*(dv/dt)$$

$$I(s)= sC*V(s)$$

Thus the impedance of the capacitor in the *s*-domain is given by:

$$Z(s) = \frac{V(s)}{I(s)} = \frac{V(s)}{sCV(s)} = \frac{1}{sC}$$

Summarizing, in the **time domain**, the circuit elements are **R, L and C** and in the **s-domain**, the circuit elements are **R, sL and (1/sC)**.

Summarizing, in the **time domain**, the circuit elements are **R, L and C** and in the **s-domain**, the circuit elements are **R, sL and (1/sC)**.

Note: that the impedance of L is $X_L = j\omega L$ and the impedance of C is $Xc = (-j/\omega C) = (1/j\omega C)$.

Thus, just replacing $j\omega$ with s gives the s-domain expressions for L and C.

(Because of this apparent association with j , s is sometimes called the **complex frequency** and the s-domain called the **complex frequency domain**.)

Kirchhoff’s laws in the s-domain

Kirchhoff’s current and voltage laws may be applied to currents and voltages in the s-domain just as they can with normal time domain currents and voltages.

To solve circuits in the s-domain using Kirchhoff’s laws the procedure is:

(i) change the time domain circuit to an s-domain circuit,

(ii) apply Kirchhoff’s laws in terms of s,

(iii) solve the equation to obtain the Laplace transform of the unknown quantity and

(iv) determine the inverse Laplace transform after rearranging into a form that can be recognized in a table of standard transforms.

Servomotors

A servo motor is defined as a linear or rotary type of actuator that provides fast precision position control for closed-loop position control applications.

The motors that are used in automatic control systems are called Servomotors. When the objective of the system is to control the position of an object then the system is called Servomechanism.

The servomotors are used to convert an electrical signal (control voltage) applied to them into an angular displacement of the shaft. They can either operate in a continuous duty or step duty depending on construction.

Servomotors

There are variety of servomotors available for control system applications. The suitability of a motor for a particular application depends on the characteristics of the system, the purpose of the system and its operating conditions

Servo Motor are also called Control motors. They are used in feedback control systems as output actuators and does not use for continuous energy conversion. The principle of the Servomotor is similar to that of the other electromagnetic motor, but the construction and the operation are different. Their power rating varies from a fraction of a watt to a few hundred watts.

Applications of the Servo Motor

The power rating of the servo motor may vary from the fraction of watts to few hundreds of watts. The rotor of servo motor have low inertia strength, and therefore they have a high speed of inertia. These motors have a high-speed response due to Low inertia and are designed with small diameters and long rotor lengths

The Applications of the Servomotor are as follows:-

- They are used in Radar system and process controller.
- Servomotors are used in computers and robotics.
- They are also used in machine tools.
- Tracking and guidance systems.

Requirements of Servomotor

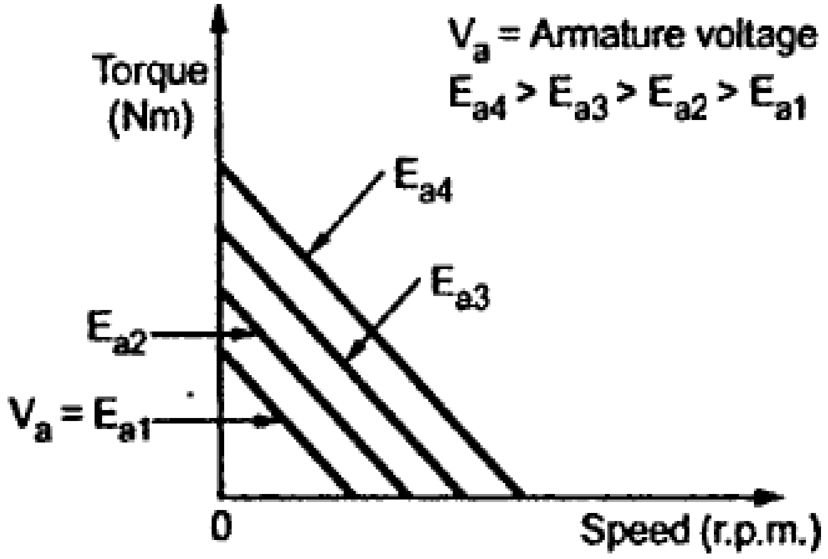
- Linear relationship between electrical control signal and rotor speed over a wide range.
- Inertia of the rotor should be low as possible.
- Fast response, Its response should be fast as possible.
- It should be easily reversible.
- It should have linear torque-speed characteristics.
- Its operation should be stable without any oscillations.
- Wide range of speed control
- Linearity of Mechanical Characteristics throughout the entire speed range.
- Low Mechanical and Electrical inertia

Servo motor	DC Motor
Servo motor works on the principle of the closed-loop control system, in which DC motor is also one of the components	A DC motor works on the principle of Lorentz’s Force Law, i.e. when a current-carrying conductor is placed in a magnetic field, it experiences a force.
Due to a closed-loop control system, servo motor would have higher precision and accuracy	This is not the case with the DC motor.
The inertia of servo motor is less	The inertia of dc motor is less
A servomotor is more suited for automation and robotics industry	The DC motors have versatile applications
Servomotor need maintenance	A DC motor is rugged in nature
Servo motors are more costly compared to DC motors	DC motors are not expensive compared to servo motors

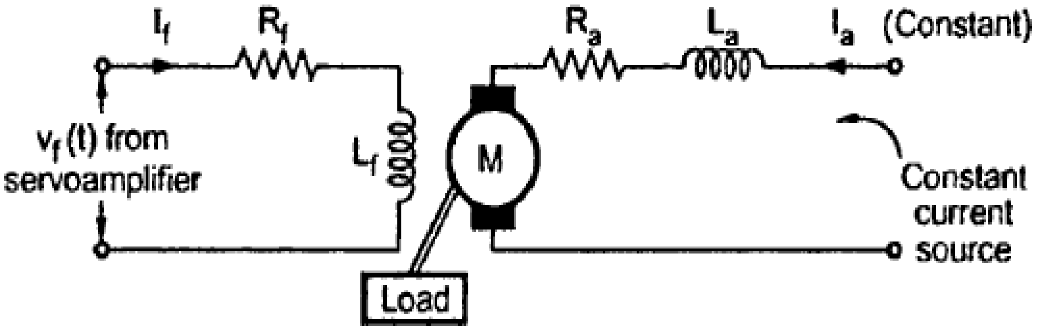
Types of Servomotors

- DC Servomotor
  - Armature controlled servomotor
  - Field controlled servomotor
- AC Servomotor

Characteristics of DC Servomotor



Field Controlled DC Servomotor

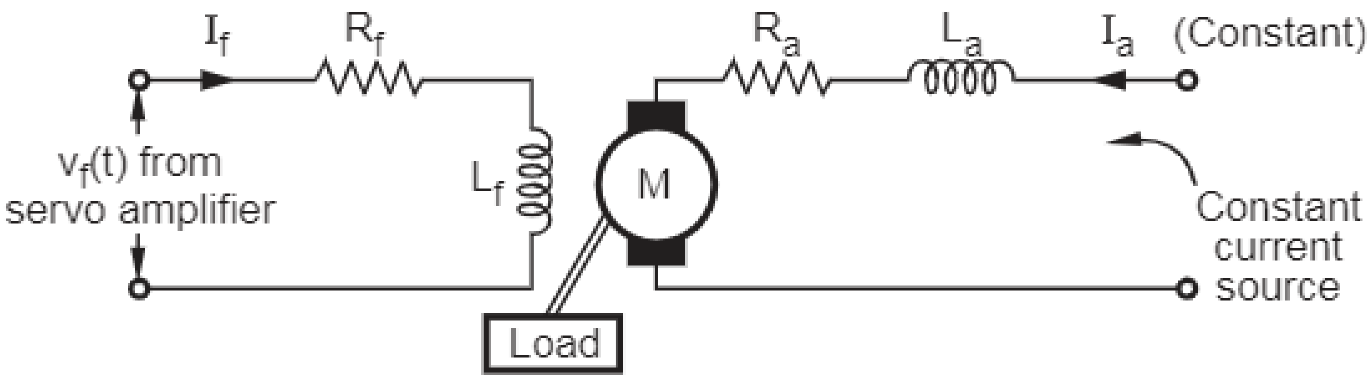


Features:

- Preferred for small rated motors
- Large time constant
- It is open loop system.
- Control circuit is simple to design.

In this method of speed control, a variable input voltage is applied to the field winding of DC motor, while keeping the armature current constant.

- In this motor, the controlled signal obtained from the servoamplifier is applied to the field winding. With the help of constant current source, the armature current is maintained constant. The arrangement is shown in the Fig.



- This type of motor has large L_f /R_f ratio where L_f is reactance and R_f is resistance of field winding. Due to this the time constant of the motor is high. This means it can not give rapid response to the quick changing control signals hence this is uncommon in practice.

The parameters are taken as

R_f = Field winding resistance.

L_f = Field winding inductance.

i_a = Armature current.

i_f = Field current.

e_f = applied field (control) voltage.

e_b = Motor back emf.

T_m = torque developed by motor.

θ = Angular displacement of motor shaft.

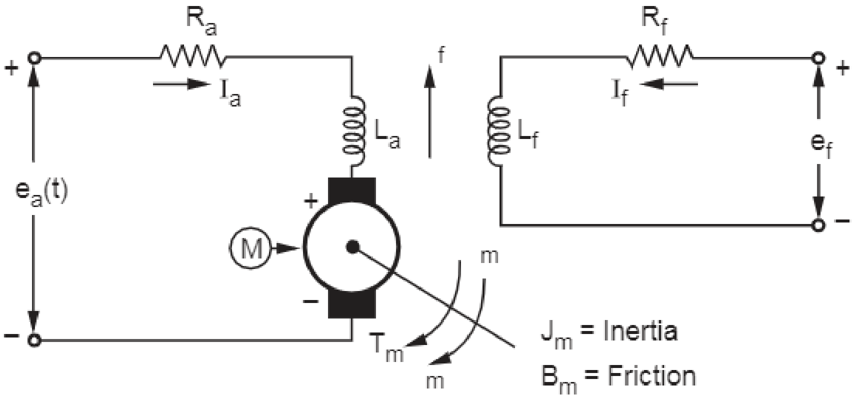
J = Equivalent moment of inertia (of load and motor) referred to motor shaft.

B = Equivalent viscous friction coefficient of motor and load referred to motor shaft.

K_T = Motor Torque Constant

K_b = Back EMF Constant

Transfer Function of Field Controlled DC motor



Assumptions :

(1) Constant armature current is fed into the motor.

(2) $\phi_f \propto I_f$. Flux produced is proportional to field current.

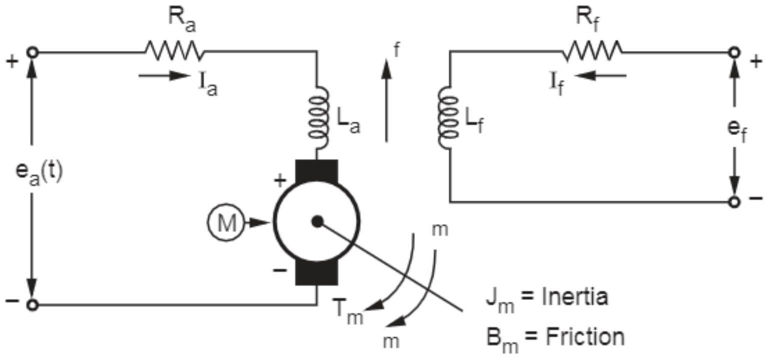
$$\therefore \quad \boxed{\phi_f = K_f I_f}$$

(3) Torque is proportional to product of flux and armature current.

$$\boxed{T_m \propto \phi I_a}$$

$$\therefore T_m = K' \phi I_a = K' K_f I_f I_a = \boxed{K_m K_f I_f} \quad \ldots (1)$$

Where $K_m = K' I_a$ = Constant



Apply Kirchhoff's law to field circuit, $L_f \dfrac{di_f}{dt} + R_f I_f = e_f$ ... (2)

- Now shaft torque T_m is used for driving load against the inertia and frictional torque.

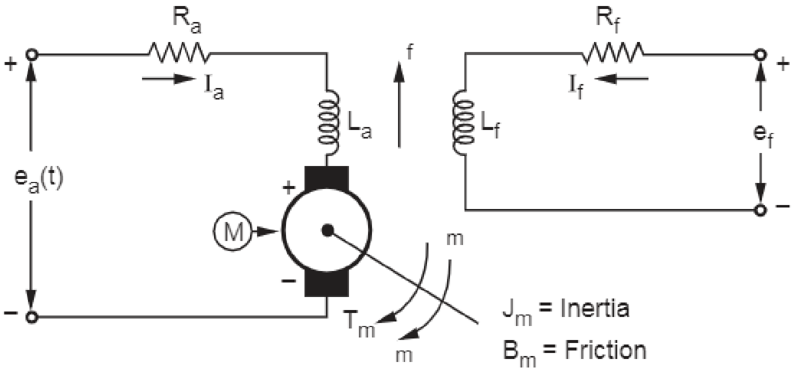
$$T_m = J_m \frac{d^2\theta_m}{dt^2} + B_m \frac{d\theta_m}{dt} \quad \text{... (3)}$$

- Finding Laplace Transforms of equations (1), (2) and (3) we get,

$$T_m(s) = K_m K_f I_f(s) \quad \text{... (4)}$$

$$E_f(s) = (sL_f + R_f)\ I_f(s) \quad \text{... (5)}$$

$$T_m(s) = J_m s^2 \theta_m(s) + B_m s\, \theta_m(s) \quad \text{... (6)}$$



- Eliminate $I_f(s)$ from equations (4) and (5)

$$T_m(s) = \frac{K_m\ K_f\ E_f(s)}{(sL_f + R_f)} \qquad \dots (\quad 7)$$

- Eliminate $T_m(s)$ from equations (6) and (7),

$$(s^2 J_m + sB_m)\,\theta_m(s) = \frac{K_m\ K_f\ E_f(s)}{(sL_f + R_f)}$$

$$\text{Input} = E_f(s) \quad \text{and}$$

$$\text{Output} = \text{Rotational displacement } \theta_m(s)$$

$$\therefore \qquad \text{Transfer function} = \frac{{}_m(s)}{E_f(s)}$$

$$\frac{{}_m(s)}{E_f(s)} = \frac{K_m\ K_f}{(J_m s^2 + sB_m)\ (R_f + sL_f)}$$

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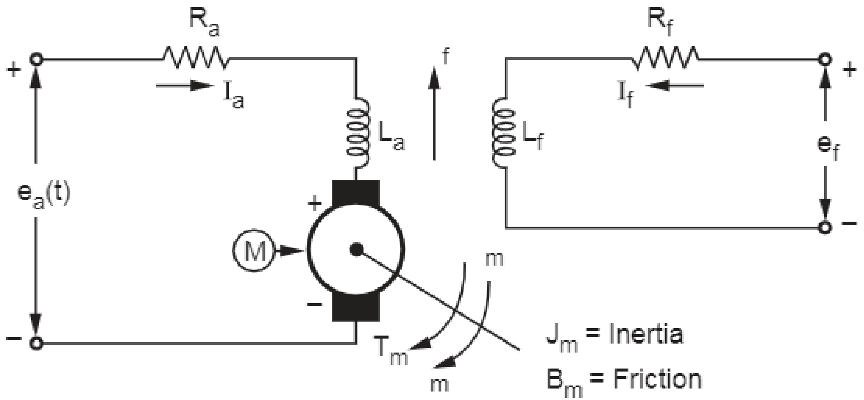


$$(s^2 J_m + sB_m)\,\theta_m(s) = \frac{K_m K_f E_f(s)}{(sL_f + R_f)}$$

$$\text{Input} = E_f(s) \quad \text{and}$$

$$\text{Output} = \text{Rotational displacement } \theta_m(s)$$

$$\therefore \qquad \text{Transfer function} = \frac{m(s)}{E_f(s)}$$

$$\frac{m(s)}{E_f(s)} = \frac{K_m K_f}{(J_m s^2 + sB_m)\,(R_f + sL_f)}$$

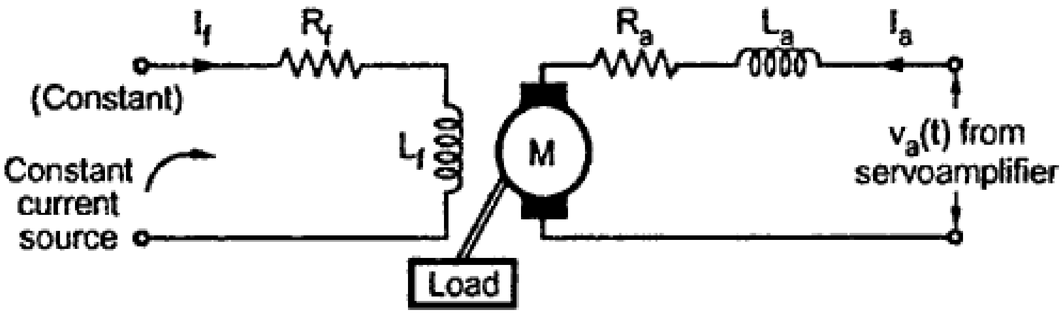
$$= \frac{K_m K_f}{sR_f B_m [1 + s\tau_m][1 + s\tau_f]}$$

Where, $\tau_m = \dfrac{J_m}{B_m}$ = Motor time constant,

$\tau_f = \dfrac{L_f}{R_f}$ = Field time constant

$$\boxed{\text{T.F.} = \frac{\theta_m(s)}{E_f(s)} = \frac{K_f}{R_f[1 + s\tau_f]} \cdot \frac{K_m}{B_m(1 + s\tau_m)} \cdot \frac{1}{s}}$$

Armature Controlled DC Servomotor

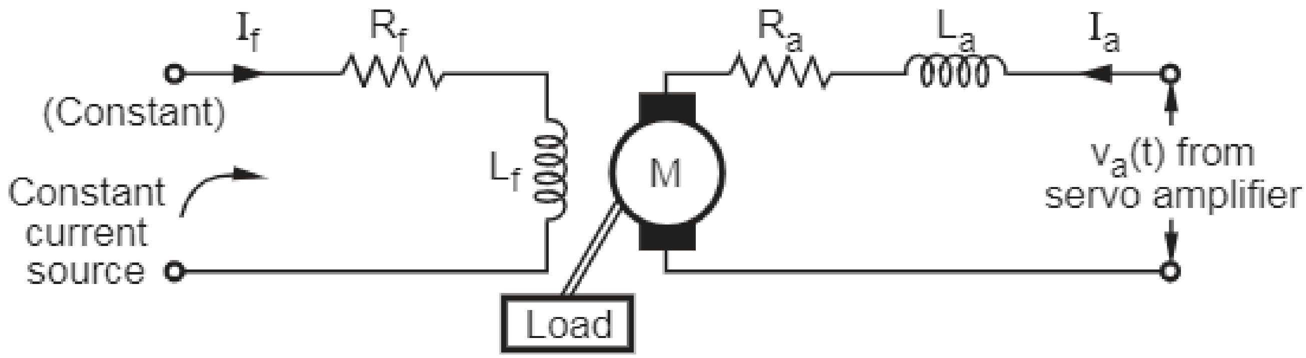


Features:

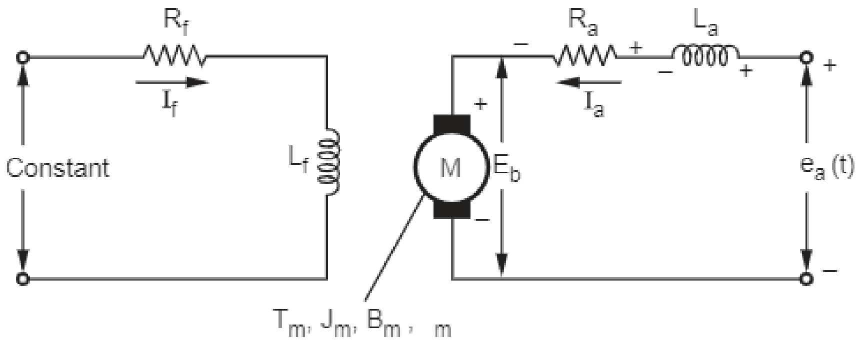
- Preferred for large rated motors
- Small time constant hence response is fast.
- It is closed loop system.
- The efficiency and overall performance is better.

In this motor, the field current is held constant and armature current is varied to control the torque.

- In this type of motor, the input voltage `V_a' is applied to the armature with a resistance of R_a and inductance L_a. The field winding is supplied with constant current I_f. Thus armature input voltage controls the motor shaft output. The arrangement is shown in the Fig.



Transfer Function of Armature Controlled DC motor



Assumptions :

(i) Flux is directly proportional to current through field winding,

$$\boxed{\phi_m = K_f I_f = \text{Constant}}$$

(ii) Torque produced is proportional to product of flux and armature current.

$$T = K'_m \phi I_a$$

$$T = K'_m K_f I_f I_a$$

(iii) Back e.m.f. is directly proportional to shaft velocity ω_m, as flux ϕ is constant.

as $\omega_m = \frac{d\theta(t)}{dt}$

$$\boxed{E_b = K_b \omega_m(s) = K_b s \theta_m(s)}$$

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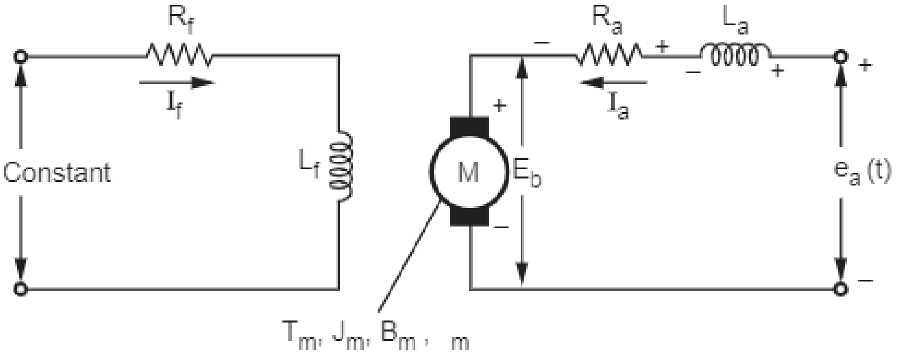


• Now shaft torque T_m is used for driving load against the inertia and frictional torque.

$$T_m = J_m \frac{d^2\theta_m}{dt^2} + B_m \frac{d\theta_m}{dt} \qquad \text{... (} \quad 3)$$

Also

Apply Kirchhoff's law to armature circuit :

$$e_a = E_b + I_a(R_a) + L_a \frac{di_a}{dt}$$

• Take Laplace transform,

$$\therefore \quad E_a(s) = E_b(s) + I_a(s)[R_a + s L_a]$$

$$\text{i.e.} \quad I_a(s) = \frac{E_a(s) - E_b(s)}{R_a + s L_a}$$

$$I_a(s) = \frac{E_a(s) - K_b s \theta_m(s)}{R_a + s L_a}$$

$$\text{Now} \quad T_m = K'_m K_f I_f I_a$$

$$T_m = K'_m K_f I_f \left\{ \frac{E_a - K_b s \theta_m(s)}{R_a + s L_a} \right\}$$

$$T_m = \{J_m s^2 + s B_m\} \theta_m(s) \qquad \text{... from equation (3)}$$

$$T_m = K'_m K_f I_f \left\{ \frac{E_a - K_b\, s\,\theta_m(s)}{R_a + s\,L_a} \right\}$$

Also $$T_m = \{J_m\, s^2 + s\,B_m\}\,\theta_m(s) \qquad \text{... from equation (3)}$$

- Equating equations of T_m,

$$\frac{K'_m K_f I_f E_a(s)}{(R_a + s\,L_a)} = \frac{K'_m K_f I_f K_b\, s\,\theta_m(s)}{(R_a + s\,L_a)} + (J_m\, s^2 + s\,B_m)\,\theta_m(s)$$

$$\therefore \frac{K'_m K_f I_f}{(R_a + s\,L_a)} E_a(s) = \left[\frac{K'_m K_f I_f K_b\, s}{(R_a + s\,L_a)} + J_m s^2 + s\,B_m\right]\theta_m(s)$$

$$\therefore \boxed{\frac{\theta_m(s)}{E_a(s)} = \frac{\dfrac{K_m}{s\,R_a\,B_m\,(1 + s\tau_m)\,(1 + s\,\tau_a)}}{1 + \dfrac{K_m \cdot sK_b}{s\,R_a\,B_m\,(1 + s\,\tau_m)\,(1 + s\,\tau_a)}} = \frac{G(s)}{1 + G(s)H(s)}}$$

where $\tau_m = J_m/B_m$ and $\tau_a = \dfrac{L_a}{R_a}$ while $K_m = K'_m K_f$

$$G(s) = \frac{K_m}{s\,R_a\,B_m\,(1 + s\,\tau_m)\,(1 + s\,\tau_a)} \quad \text{and} \quad H(s) = sK_b$$

$$T_m = K'_m K_f I_f \left\{ \frac{E_a - K_b\, s\, \theta_m(s)}{R_a + s\, L_a} \right\}$$

Also $$T_m = \{J_m s^2 + s\, B_m\}\theta_m(s) \qquad \text{... from equation (3)}$$

- Equating equations of T_m,

$$\frac{K'_m K_f I_f E_a(s)}{(R_a + s\, L_a)} = \frac{K'_m K_f I_f K_b\, s\, \theta_m(s)}{(R_a + s\, L_a)} + (J_m s^2 + s\, B_m)\,\theta_m(s)$$

$$\therefore \frac{K'_m K_f I_f}{(R_a + s\, L_a)} E_a(s) = \left[\frac{K'_m K_f I_f K_b\, s}{(R_a + s\, L_a)} + J_m s^2 + s\, B_m \right] \theta_m(s)$$

$$\therefore \boxed{\frac{\theta_m(s)}{E_a(s)} = \frac{\dfrac{K_m}{s\, R_a\, B_m\, (1 + s\tau_m)(1 + s\,\tau_a)}}{1 + \dfrac{K_m \cdot sK_b}{s\, R_a\, B_m\, (1 + s\,\tau_m)(1 + s\,\tau_a)}} = \frac{G(s)}{1 + G(s)H(s)}}$$

where $\tau_m = J_m / B_m$ and $\tau_a = \dfrac{L_a}{R_a}$ while $K_m = K'_m K_f$

$$G(s) = \frac{K_m}{s\, R_a\, B_m\, (1 + s\,\tau_m)(1 + s\,\tau_a)} \quad \text{and} \quad H(s) = sK_b$$

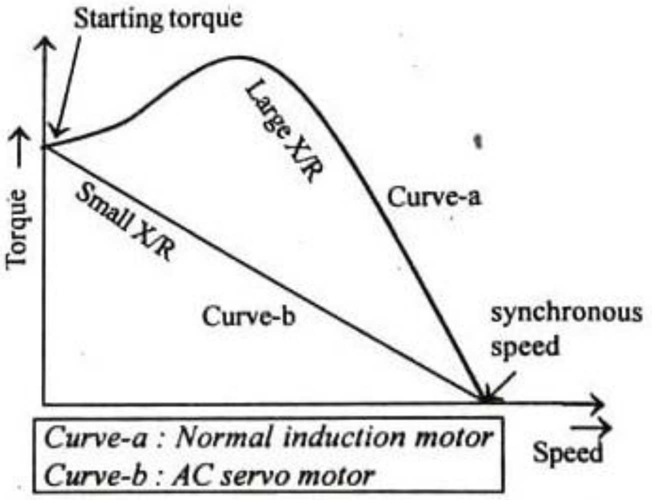
AC servomotor

An AC servomotor is basically a two-phase induction motor except for certain special design features. A two-phase servomotor differs in the following two ways from a normal induction motor

1. The rotor of the AC servomotor is built with high resistance, so that its X/R (Inductive reactance/Resistance) ratio is small which results in linear speed torque characteristics. (But conventional induction motors will have high value of X/R ratio which results in high efficiency and non-linear speed-torque characteristics).

2. The excitation voltage applied to two stator windings should have a phase difference of 90°

The speed-torque characteristics of normal induction motor (curve-a) and AC servomotor(curve-b) are shown in fig.

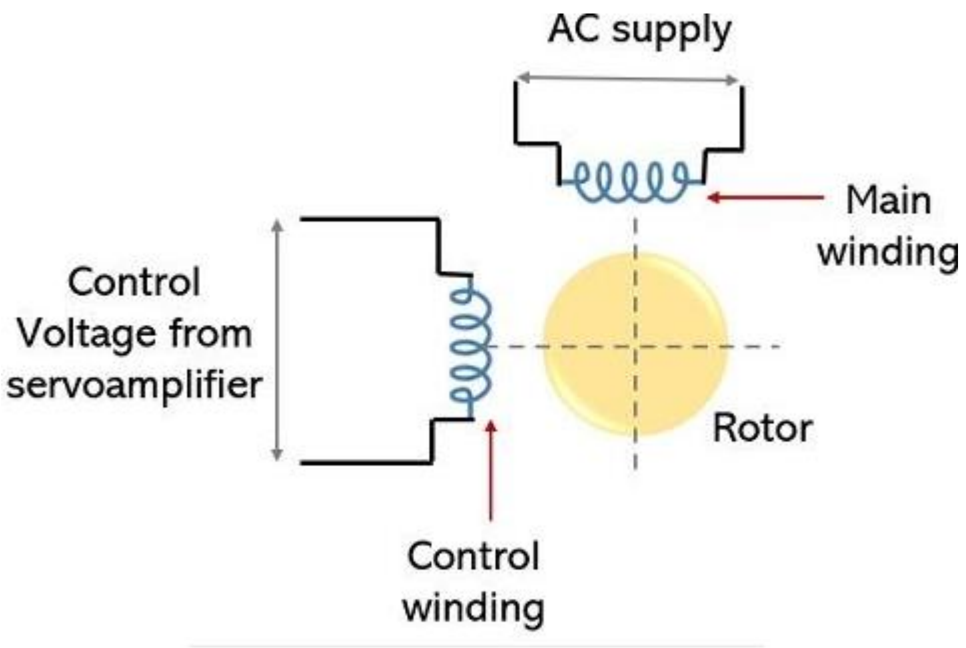


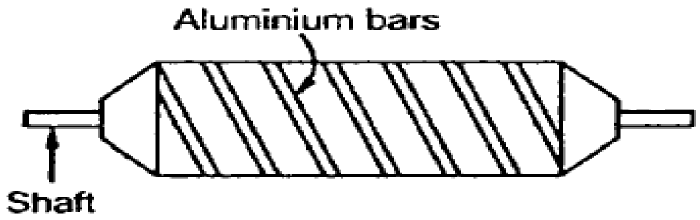
Difference between AC servomotor and Two Phase induction motor

1. The AC servomotor has low value of X/R to achieve linear speed-torque characteristics. But conventional induction motor will have large values of X/R for higher efficiency.
2. The AC servomotor has low inertia rotor. The inertia of the rotor is reduced by reducing the diameter or by drag-cup construction.

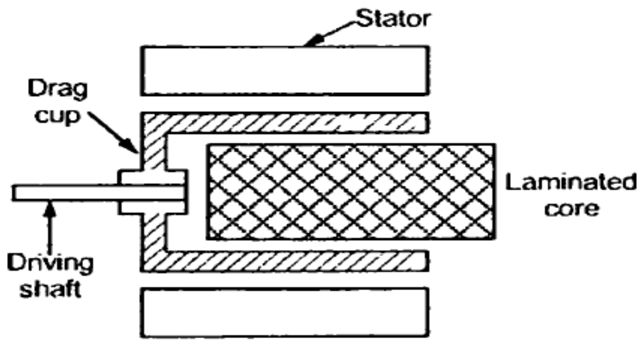
Types of rotors of AC servomotor are squirrel-cage rotor and drag-cup rotor

AC servomotor





Squirrel-cage rotor

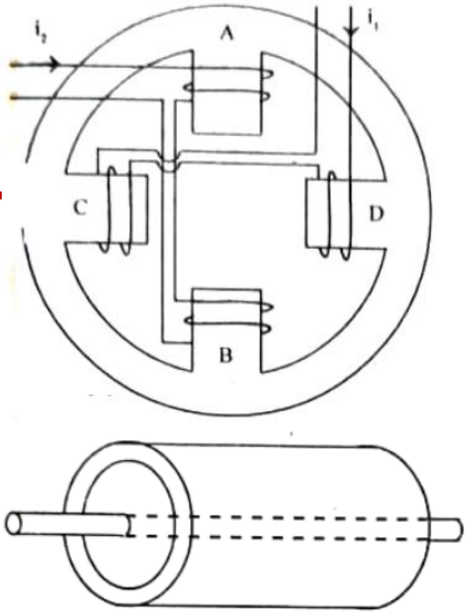


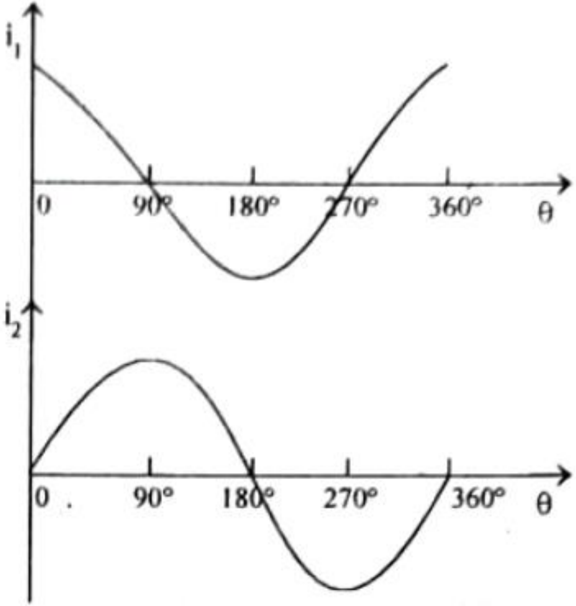
Drag-cup rotor

Types of rotors of AC servomotor are squirrel-cage rotor and drag-cup rotor

CONSTRUCTION OF AC SERVOMOTOR

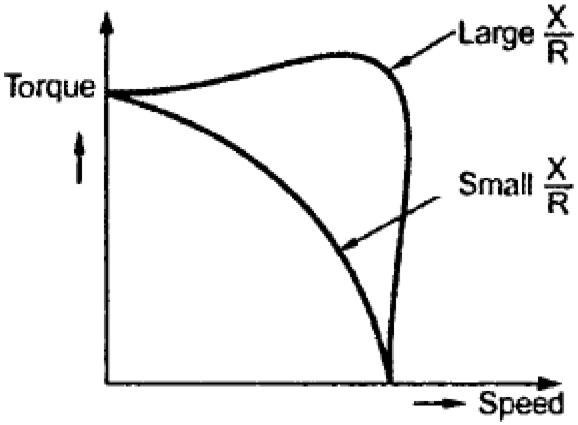
The AC servomotor is basically a two phase induction motor with some special design features. The stator consists of two pole-pairs (A-B and C-D) mounted on the inner periphery of the stator, such that their axes are at an angle of 90°

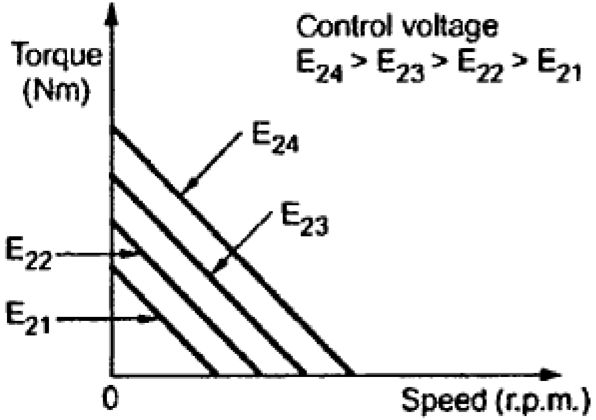




Exciting Current

Characteristics of AC servomotors





Compare of DC servomotor and AC servomotor

Sl.No	DC Servo motor	AC Servomotor
1	Higher power output	Relatively lesser power output than a DC servomotor of same size
2	Characteristics are linear	Characteristics are Non- linear
3	Fast response due to low electrical and mechanical time constant.	The response is relatively slower than DC servomotors due to higher values of time constants.
4	Suitable for large power applications	Suitable for low power applications.

AC Servomotor

- Features:
  - Light in weight
  - Robust construction
  - Reliable and stable operation.
  - Smooth and noise free operation
  - Large torque to weight ratio
  - Less X/R ratio
  - Maintenance free

Transfer Function of A.C. Servomotor

- The various **approximations** to derive transfer function are,

(i) A servomotor rarely operates at high speeds. Hence for a given value of control voltage, T ∝ N characteristics are perfectly linear.

(ii) In order that T ∝ N characteristics are directly proportional to voltage applied to its control phase, we assume T ∝ N characteristics are straight lines and equally spaced.

- Torque at any speed 'N' is,

$$T_m = K_{tm} E_{2t} + m \frac{d\theta_m}{dt} \quad \ldots (1)$$

- where, $\frac{d\theta_m}{dt}$ is speed of motor.

- If load consists inertia J_m and friction B_m we can write,

$$T_m(s) = J_m s^2 \theta_m + B_m s \theta_m \quad \ldots (2)$$

- Now Laplace transform of equation (1) is

$$T_m(s) = K_{tm}\, E_2(s) + m\, s\, \theta_m(s) \qquad \ldots (\quad 3)$$

- Equating equations (2) and (3)

$$\therefore\ K_{tm}\, E_2(s) + m\, s\, \theta_m(s) = J_m\, s^2\, \theta_m(s) + B_m\, s\, \theta_m(s)$$

$$\therefore \qquad \frac{\theta_m(s)}{E_2(s)} = \frac{K_{tm}}{s\,(s\,J_m - m + B_m)}$$

$$= \frac{K_{tm}}{s\,(B_m - m)\left[1 + \frac{s\,J_m}{(B_m - m)}\right]}$$

$$\therefore \qquad \boxed{\frac{\theta_m(s)}{E_2(s)} = \frac{K_m}{s\,(1 + \tau_m\, s)}}$$

where $\qquad K_m = \dfrac{K_{tm}}{B_m - m}$

and $\qquad \tau_m = \dfrac{J_m}{B_m - m}$

Gear Trains

A gear train is a mechanical device that transmits energy from one part of a system to another in such a way that force, torque, speed and displacement may be altered. The inertia and friction of the gears are neglected in the ideal case. Consider a gear system as shown in the Fig.

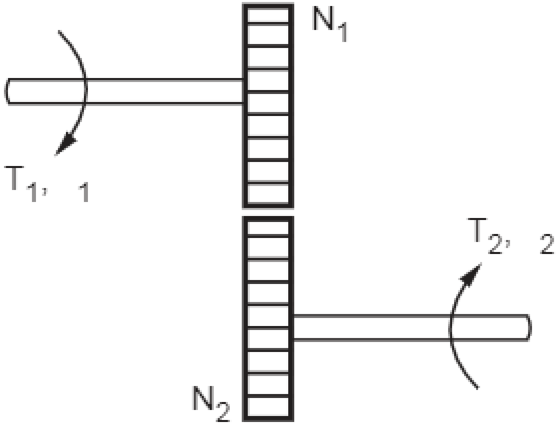
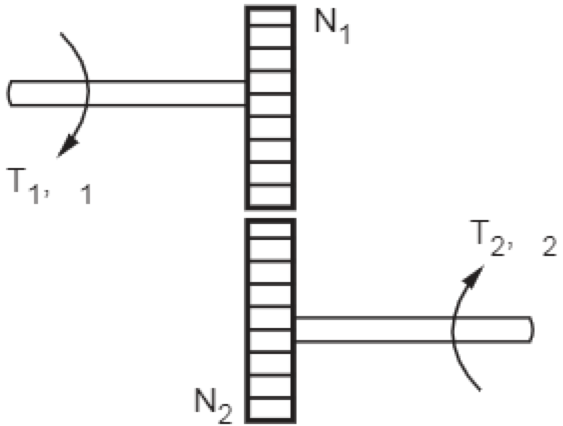


Fig. Gear system

The number of teeth on the surface of the gears is proportional to the radii r_1 and r_2 of the gears.

i.e. $$\boxed{r_1 N_2 = r_2 N_1}$$



T = Applied torque

θ_1 , θ_2 = Angular displacements

T_1 , T_2 = Torque transmitted to gears

J_1 , J_2 = Inertia of gears

N_1 , N_2 = Number of teeth

B_1 , B_2 = Friction coefficients.

- The distance travelled along the surface of each gear is same.

i.e. $$\boxed{\theta_1 \; r_1 = \theta_2 \; r_2}$$

- The work done by one gear is same as the other.

i.e. $$T_1 \theta_1 = T_2 \theta_2$$

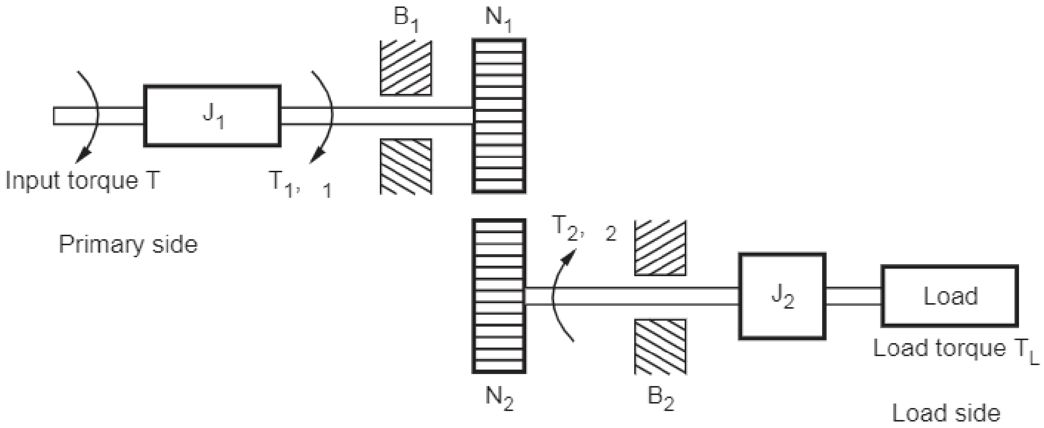
$\therefore$ we can say $$\boxed{\frac{T_1}{T_2} = \frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2}}$$

Remarks :

1) The numbers of teeth N are proportional to the radius r of a gear.
2) The distance travelled on each gear is same.
3) Work done = Tθ by each gear is same.

Gear Train with Inertia and Friction

In practice, gears do have inertia and friction which cannot be neglected. Consider such practical gear arrangement connected to the load, shown in the Fig.



T = Applied torque

θ_1, θ_2 = Angular displacements

T_1, T_2 = Torque transmitted to gears

J_1, J_2 = Inertia of gears

N_1, N_2 = Number of teeth

B_1, B_2 = Friction coefficients.

- Torque equation of side 1 is,

$$T = J_1 \frac{d^2\theta_1(t)}{dt^2} + B_1 \frac{d\theta_1(t)}{dt} + T_1(t) \quad \ldots (1)$$

- Torque equation of side 2 is,

$$T_2 = J_2 \frac{d^2\theta_2(t)}{dt^2} + B_2 \frac{d\theta_2(t)}{dt} + T_L(t) \quad \ldots (2)$$

Now $\frac{T_1}{T_2} = \frac{N_1}{N_2} = \frac{\theta_2}{\theta_1}$ i.e. $T_2 = \frac{N_2}{N_1} T_1$

- Substituting in equation (2)

$$\therefore \quad \frac{N_2}{N_1} T_1 = J_2 \frac{d^2 \theta_2}{dt^2} + B_2 \frac{d\theta_2}{dt} + T_L$$

$$\therefore \quad T_1 = \frac{N_1}{N_2} J_2 \frac{d^2\theta_2}{dt^2} + \frac{N_1}{N_2} B_2 \frac{d\theta_2}{dt} + \frac{N_1}{N_2} T_L \quad \dots (3)$$

- Substituting value of T_1 in equation (1)

$$T = J_1 \frac{d^2 \theta_1}{dt^2} + B_1 \frac{d\theta_1}{dt} + \frac{N_1}{N_2} J_2 \frac{d^2 \theta_2}{dt_2} + \frac{N_1}{N_2} B_2 \frac{d\theta_2}{dt} + \frac{N_1}{N_2} T_L$$

Substituting $\theta_2 = \frac{N_1}{N_2} \theta_1$

Substituting $\theta_2 = \frac{N_1}{N_2}\theta_1$

$$\therefore T = J_1 \frac{d^2\theta_1}{dt^2} + B_1 \frac{d\theta_1}{dt} + \frac{N_1}{N_2} J_2 \frac{N_1}{N_2} \frac{d^2\theta_1}{dt_2} + \frac{N_1}{N_2} \cdot B_2 \frac{N_1}{N_2} \frac{d\theta_1}{dt} + \frac{N_1}{N_2} T_L$$

$$\therefore T = \left[J_1 + \left(\frac{N_1}{N_2}\right)^2 J_2\right] \frac{d^2\theta_1}{dt^2} + \left[B_1 + \left(\frac{N_1}{N_2}\right)^2 B_2\right] \frac{d\theta_1}{dt} + \frac{N_1}{N_2} T_L$$

$\therefore$ J_{1e} = Equivalent inertia referred to primary side

$$\boxed{J_{1e} = J_1 + \left(\frac{N_1}{N_2}\right)^2 J_2}$$

B_{1e} = Equivalent friction referred to primary side

$$\boxed{B_{1e} = B_1 + \left(\frac{N_1}{N_2}\right)^2 B_2}$$

$$\boxed{T = J_{1e} \frac{d^2\theta_1}{dt_2} + B_{1e} \frac{d\theta_1}{dt} + \left(\frac{N_1}{N_2}\right) T_L}$$

B_{1e} = Equivalent friction referred to primary side

$$\boxed{B_{1e} = B_1 + \left(\frac{N_1}{N_2}\right)^2 B_2}$$

$$\boxed{T = J_{1e}\frac{d^2\theta_1}{dt_2} + B_{1e}\frac{d\theta_1}{dt} + \left(\frac{N_1}{N_2}\right)T_L}$$

$$\boxed{\left(\frac{N_2}{N_1}\right)T = J_{2e}\frac{d^2\theta_2}{dt^2} + B_{2e}\frac{d\theta_2}{dt} + T_L}$$

where $J_{2e} = J_2 + \left(\frac{N_2}{N_1}\right)^2 J_1$ and

$$B_{2e} = B_2 + \left(\frac{N_2}{N_1}\right)^2 B_1$$

Synchro Pair

Synchro is a device used to convert an angular motion to an electrical signal or vice versa. Its works on the principal of a rotating transformer

The trade names for synchronous are Selsyn, Autosyn and Telesyn. Basically they are electro mechanical devices or electromagnetic transducer which produces an output voltage depending upon angular position of the rotor.

A synchro pair is a system formed by interconnection of the devices : synchro transmitter and synchro control transformer.

A synchro pair is used to either transmit an angular motion from place to another or employed to produce an error voltage proportional to the difference between the angular motions.

A Synchro system is formed by interconnection of the devices called the Synchro Transmitter and the synchro control transformer. They are also called as synchro pair.

The synchro pair measures and compares two angular displacements and its output voltage is approximately linear with angular difference of the axis of both the shafts.

They can be used in the following two ways.

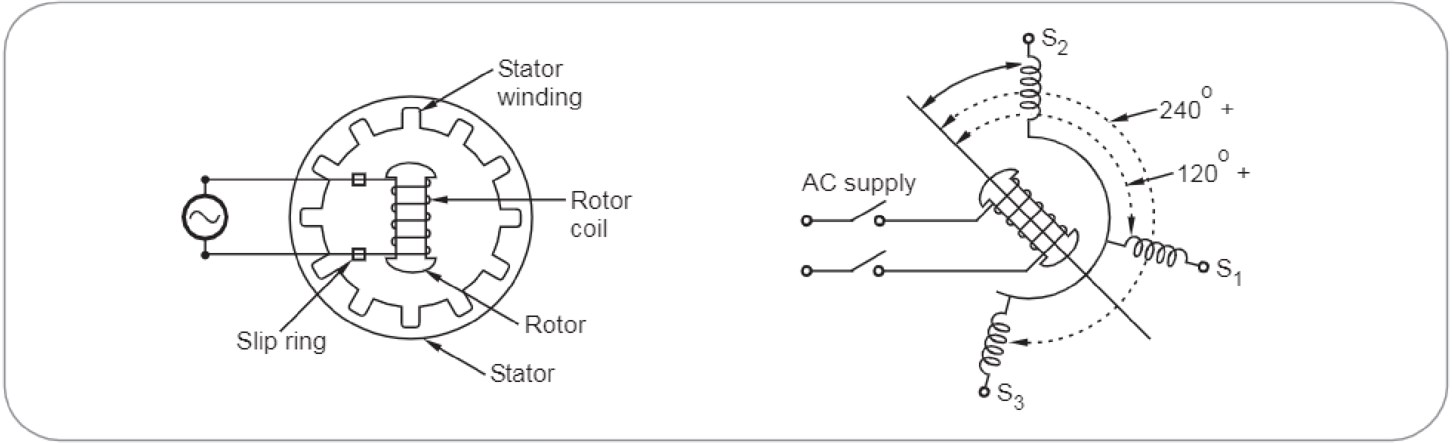
i. To control the angular position of load from a remote place / long distance.

ii. For automatic correction of changes due to disturbance in the angular position of the load.

Synchro Transmitter

This is a basic synchro unit. It's construction is similar to that of 3 phase alternator. • The stator which is stationary part is made up of laminated steel. This part is slotted to accommodate a balanced three phase winding. The stator windings are star connected which are usually of concentric coil type structure.

The rotor of Synchro Transmitter which is rotating part is a salient pole, dumb-bell shaped magnet with a single winding. Schematic diagram is as shown in Fig A single phase AC voltage is applied to the rotor throug



Synchro Control Transformer

Synchro control transformer is an electro mechanical device which produces a single phase voltage whose magnitude is proportional to the sine of the angle of rotation of its rotor with respect to stator magnetic field.

In control transformer, the induced emf in the rotor is used as an output signal(ERROR SIGNAL)

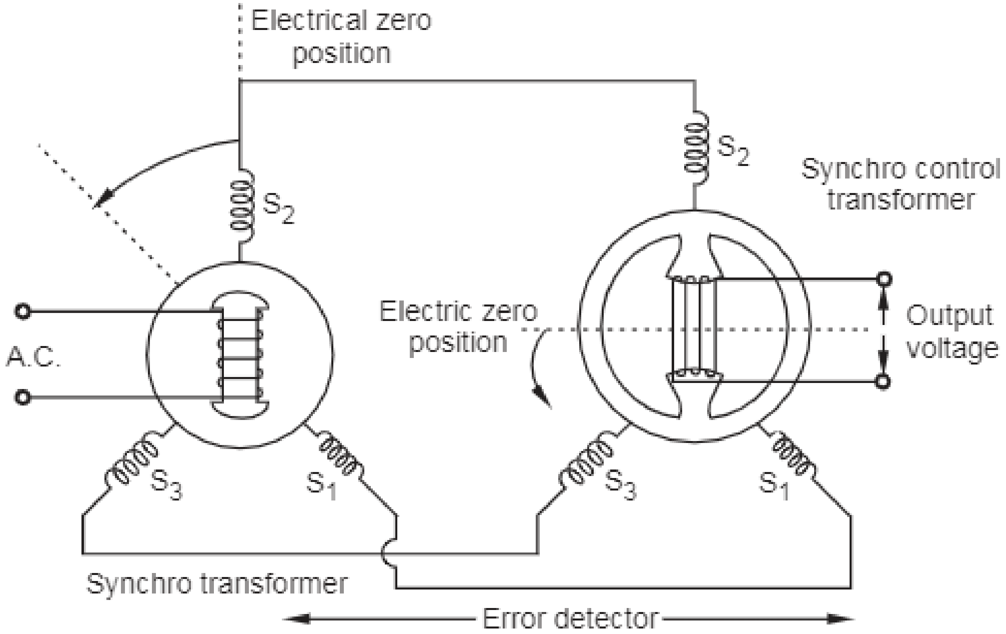
Principle of operation of synchro control transformer is same as that of synchro transmitter. Rotor of synchro control transformer is cylindrical type so that the airgap flux is uniformly distributed around the rotor.

Error Detector using Synchros

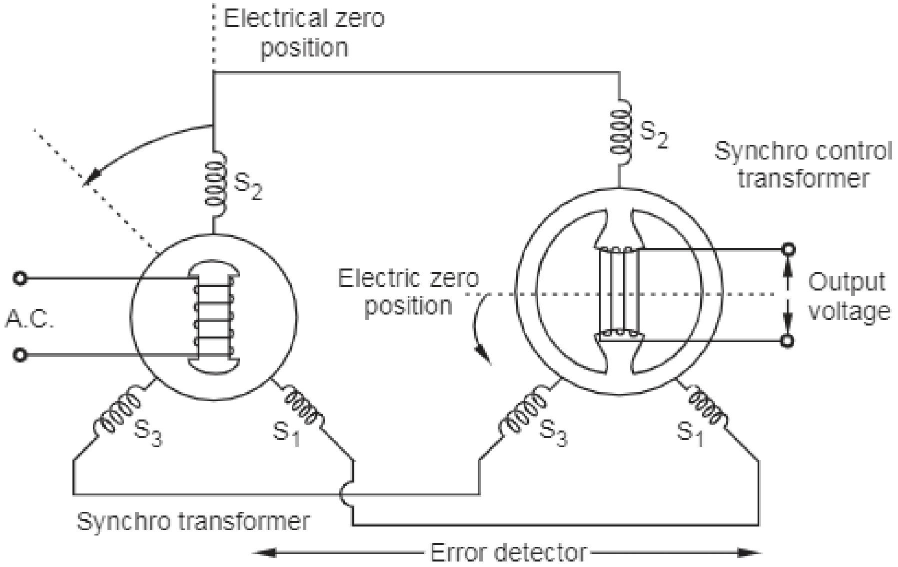
The function of error detector is to convert the difference of two shaft positions into an electrical signal so to use synchros as error detector along with synchro transmitter one more component is required called synchro control transformer. An error detector involves both synchros synchro transmitter as well synchro control transformer.

Error Detector using Synchros

The Fig. shows schematic diagram of synchro error detector in which second component is synchro control transformer. The output of synchro transmitter is given to synchro control transformer.



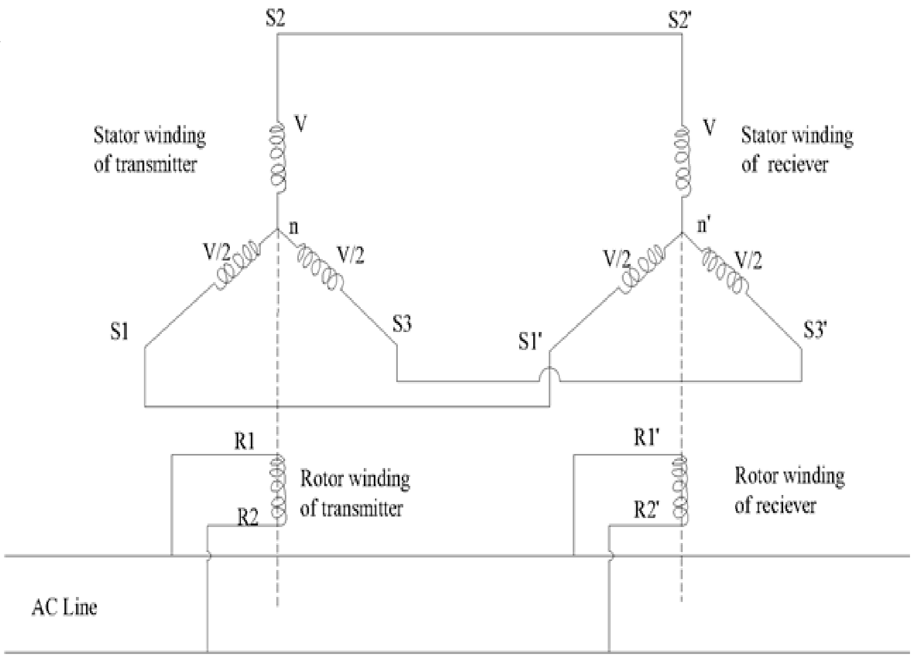
Error Detector using Synchros



When the rotor positions of the two synchros are in perfect alignment, the voltage generated across the terminals of the rotor of control transformer is zero. This position is called Electrical zero position of control transformer.

What is Electrical Zero of synchro

The electrical zero of a synchro transmitter is a position of rotor at which one of the coil to coil voltage is zero. Any angular motion of the rotor is measured with respect to the electrical zero position of the rotor. For the arrangement shown in fig, the coil S2, will have maximum emf induced in it and the coil to coil voltage S13, will be zero and the rotor is in electrical zero position.



What is Null position in Synchro?

The Null position of a synchro control transformer in a servo system is defined as, the position of its rotor for which the output voltage on the rotor winding is zero, with the transmitter in its electrical zero position.

What is aligned position of a synchro pair?

In the aligned position of a synchro pair, the transmitter rotor will be in electrical zero position and the control transformer rotor will be in null position. The angular separation of both rotor axis in aligned position is 90°. The error signal is zero in the aligned position.

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